Short-Run Dynamics in a Search-Theoretic Model of Monetary Exchange*

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Abstract

Staring from the seminal work of Kiyotaki and Wright (1989), a large body of research has been dedicated to develop micro-founded monetary models. However, due to technical difficulties, the short-run implications of such models have not been fully explored. This paper contributes to this literature by analyzing the short-run dynamics of a random-matching model of money in which agents are subject to idiosyncratic liquidity shocks as well as aggregate monetary shocks. Monetary policy has redistributive effects and persistent effects on output and prices: aggregate shocks will propagate and diffuse gradually as the money distribution adjusts over time. The model is used to study the short-run effects of unanticipated inflation shocks and inflation target changes. We find that monetary policy can have persistent effects on output, prices and welfare, even in the absence of exogenous nominal rigidities. Our result highlights the fact that explicitly modelling the micro-foundation of money is important not only for long-run steady state analysis, but also important for understanding the short-run effects of monetary policy.

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1 Introduction

Many economists agree that it is important to develop internally consistent economic models for studying short-run and long-run effects of monetary policy. Staring from the seminal work of Kiyotaki and Wright (1989), a large body of research has been dedicated to develop micro-founded

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monetary models. A successful class of such models has been the search-theoretical monetary literature using a random matching model. By explicitly modelling the role for money as a medium of exchange, this literature has been successful in addressing a large number of theoretical and monetary policy questions. However, due to technical difficulties, the short-run implications of such models have not been fully explored.

The key reason is that, in this class of models, the role of money is supported by introducing matching, information and commitment frictions. The presence of these frictions, however, tends to imply a non-degenerate distribution of money holdings. Keeping track of the distribution is important because, unlike in standard models with a centralized market, agents in a matching model are subject to random pairwise exchanges and they need to know, not only one market price, but the whole distribution to make their consumption, saving and portfolio decision. Since the distribution responds endogenously to aggregate shocks, it is technically challenging to keep track of the evolution of distribution.

Owing to this technical difficulty, existing work has typically either focused only on long-run (steady-state) analysis (e.g., Lagos and Wright 2005, Molico 2006) or need to assume away the distributional issues in short-run analysis by making some extreme simplifying assumptions. For example, some papers assume that money is an indivisible object (e.g., Wallace 1997), while others assume some form of perfect insurance against the idiosyncratic trading risk that naturally arises in such models (e.g., Shi 1999, Arouba and Wright 2003).[1] These restrictive assumptions eliminate some of the most interesting features of standard search models of money – features that could have important implications for the short-run and long-run effects of monetary policy. As pointed out by Lagos and Wright (2002) their approach "miss the way changes in parameters or policy variables might affect an endogenous distribution and how this could affect other variables, including welfare".[2] Therefore, while existing literature has highlighted the importance of distributional and propagational effects of monetary policy, the short-run dynamics of this class of models is still not fully understood.[3]

[1]In particular, Shi (1999) assumes agents have access to perfect insurance by belonging to a large household, while Aruoba and Wright (2005), Telyukova and Visschers (2008) assume agents perfectly undo their idiosyncratic trading histories by having a quasi-linear preference.

[2]One example is the short-run distributional effects of monetary policy. A recent empirical work finds that expansionary monetary policy shocks can generate significant redistribution and reduce inequality in the U.S. (Coibion et. al., 2012).

[3]For example, Berentsen et al. (2005), Boela and Camera (2009), Chiu and Molico (2010), Dressler (2009), and

Our paper is the first work to tackle this problem and contribute to the search-theoretical literature by studying the distributional effects and the internal propagation mechanism of a random-matching model of money. Specifically, we develop a search model in which agents are subject to both idiosyncratic and aggregate shocks. Agents are unable to insure against such risk due to market incompleteness implied by the frictions that generate the role of money. In this environment, heterogenous idiosyncratic histories imply a non-degenerate distribution of money holdings which becomes a state variable summarizing the past histories. As a result, monetary policy has redistributive effects and persistent effects on output and prices: aggregate shocks will propagate and diffuse gradually as the money distribution adjusts over time.

We solve the model to study the dynamic responses to monetary policy including the effects of unanticipated inflation shocks and the effects of changing long-run inflation target. We contribute to the literature by uncovering a novel feature of random-matching models: monetary policy can have persistent effects on output, prices and welfare, even in the absence of exogenous nominal rigidities. Our result highlights the fact that explicitly modelling the micro-foundation of money is important not only for long-run analysis, but also important for understanding the short-run effects of monetary policy.[4] As is well known, solving models with heterogeneous agents and aggregate uncertainty constitutes a technical challenge since the set of state variables contains the cross-sectional distribution of agent's characteristics, which is a time-varying infinite dimensional-object in the presence of aggregate uncertainty. We contribute to the literature by introducing a recently developed computational method to analyze the distributional effects of monetary policy in random-matching models. This is particularly useful in a search-theoretical model of money since the assumption of random matching implies that the agents must form an expectation regarding the potential gains from trade in order to solve their problems. The algorithm developed by Algan, Allais, and Den Haan (2008) allows us to compute such expectation using quadrature technique without introducing a significant approximation error.

The rest of the paper is organized as follows. Section 2 describes the model environment. Section 3 defines a recursive equilibrium. Section 4 defines the approximate economy. Section 5 discusses

Williamson (2006) study non-degenerate distribution in random matching models. Menzio, Shi and Sun (2010) analyze a different environment with directed search.

[4]Our work is also related to Algan and Ragot (2010) who study redistributive effects of monetary policy in an incomplete market model with centralized trading and money-in-utility.

in detail the numerical algorithm used to compute the recursive equilibria of the approximate economy. In section 6, we parameterize and characterize the model. Section 7 concludes the paper.

2 The Model

Environment

In order to study short-run effects of monetary policy in a search model of money, we extend Molico (2006) by introducing monetary growth shocks. Time is discrete. There is a [0,1] continuum of infinitely lived agents who specialize in the consumption and production of perfectly divisible nonstorable goods. Specialization is such that no agent consumes the good he produces; each agent consumes goods that are produced by a fraction x of the population, and each agent produces goods that are consumed by a fraction x of the population. For simplicity, no two agents consume the output of each other; that is, there is never a double coincidence and hence there is no direct barter.[5]

Agents produce goods by exerting labor effort h. The production function is assumed to be linear, given by $q_t = h_t$. Agents derive utility $u(q)$ from consuming q units of a good, and disutility h from exerting labor effort h. Assume $u(\cdot)$ is C^2, strictly increasing, strictly concave, and satisfies $u(0) = 0$. Let $0 < \beta < 1$ denote the discount factor.

In this economy, there is an additional, perfectly divisible, and costlessly storable object that cannot be produced or consumed by any private individual, called *fiat money.* Agents can hold any non-negative amount of money $\hat{m} \in \mathbb{R}_+$. The money stock at the beginning of period t is denoted M_t. In what follows we express all nominal variables as fraction of the beginning of the period money supply (before any of the current period's money transfers which we will describe below), i.e. $m_t \equiv \hat{m}_t/M_t$. Let $\lambda_t : \mathfrak{B}_{\mathbb{R}_+} \to [0, 1]$ denote the probability measure associated with the distribution of money holdings (as a fraction of the beginning of period money supply) at the beginning of period t, where $\mathfrak{B}_{\mathbb{R}_+}$ denotes the Borel subsets of $\mathbb{R}_+$. The money supply grows at rate $\mu_t = G(\lambda_t) + \epsilon_{\mu t}$, $M_{t+1} = \mu_t M_t$, where $G(\lambda_t) : \Lambda \to \mathbb{R}_+$ is a time-invariant monetary policy rule and ϵ^{μ}_t is an i.i.d. money growth shock. Let $\epsilon_{\mu t} \in \{-\Delta, 0, \Delta\}$ with $\Pr(\epsilon_\mu = -\Delta) = \Pr(\epsilon_\mu = \Delta) = \tau < \frac{1}{2}$. Newly

[5] For money to be valued it is only required that in some meetings there is no double coincident of wants. For simplicity, we focus on purely monetary trades and, by assumption, preclude the possibility of barter.

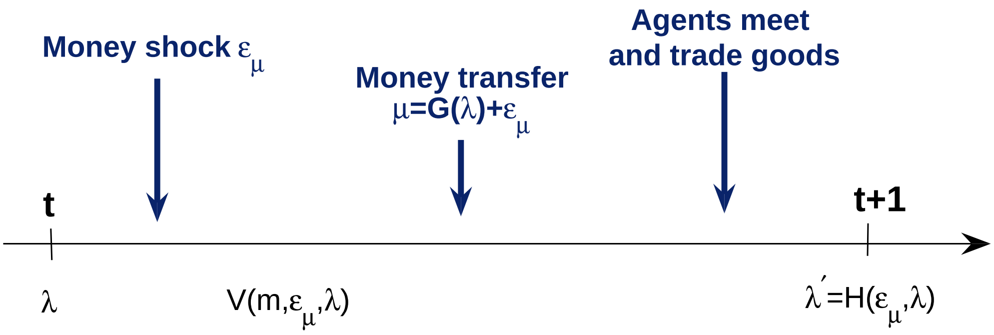

Figure 1: Time-line

printed money is injected in the economy at the beginning of the period via lump-sum transfers to all agents. (Figure 1)

Meetings

Each period, agents are randomly bilaterally matched. Given the pattern of specialization, in a given meeting, an individual will be a potential buyer with probability x. Also, with probability x the agent will be a potential seller. By assumption, there is never double-coincidence-of-wants meetings. When individuals meet in a single coincidence meeting they bargain over the amount of money and goods to be traded. Let $q(m_b, m_s; \epsilon_{\mu t}, \lambda_t) \geq 0$ denote the amount of good and $d(m_b, m_s; \epsilon_{\mu t}, \lambda_t) \geq 0$ the amount of money to be determined by the bargaining process at date t between a buyer with money holdings m_b and a seller with m_s, when the current state of the economy is described by $(\epsilon_{\mu t}, \lambda_t)$. We assume that when the two agents meet the buyer makes a take-it-or-leave-it offer to the seller, extracting all of the seller's expected surplus. This concludes the description of the environment. In what follows, we will gradually build towards the definition of an equilibrium.

3 Recursive Equilibrium

In this section we define a recursive equilibrium for this economy. A key element in such definition is the law of motion of the aggregate state of the economy. The aggregate state is given by (ϵ_μ, λ): the current monetary shocks, and the current probability measure over money holdings. The part of the law of motion that concerns ϵ_μ is exogenous, and is described by the i.i.d. process governing ϵ_μ. The part that concerns updating λ is denoted H, $\lambda' = H(\epsilon_\mu, \lambda)$, where prime denotes a variable in the following period. For the individual agent, the relevant state variable is his holdings of money and the aggregate state: $(m, \epsilon_\mu, \lambda)$. Agents take as given the law of motion of the distribution of money holdings, the process governing the monetary policy shocks, and the monetary policy rule.

Let V denote the value of money and at the beginning of the period before monetary transfers but after observing the current period's aggregate shock ϵ_μ. The following equation describes the value function of an agent with money holding m given the aggregate state (ϵ_μ, λ).[6]

$$\begin{aligned} V(m;\epsilon_\mu,\lambda) \quad = \quad & x\beta \int_0^\infty u[q(m,m_s;\epsilon_\mu,\lambda)] + E_{\epsilon'_\mu}\left\{ V\left[\frac{m+\mu-1-d(m,m_s;\epsilon_\mu,\lambda)}{\mu};\epsilon'_\mu,\lambda'\right]\right\}\lambda(dm_s) \\ & + x\beta \int_0^\infty -q(m_b,m;\epsilon_\mu,\lambda)] + E_{\epsilon'_\mu}\left\{ V\left[\frac{m+\mu-1+d(m_b,m;\epsilon_\mu,\lambda)}{\mu};\epsilon'_\mu,\lambda'\right]\right\}\lambda(dm_b) \\ & +(1-2x)\beta E_{\epsilon'_\mu} V\left[\frac{m+\mu-1}{\mu};\epsilon'_\mu,\lambda'\right]. \end{aligned}$$

where E denotes the expectations operator and a prime superscript a next period's variable.

The first line of the RHS describes the situation when an agent is a buyer, which happens with probability x. When a buyer meets a seller with money holding m_s, he will consume $q(m, m_s; \epsilon_\mu, \lambda)$, and reduce his money holding by $d(m, m_s; \epsilon_\mu, \lambda)$. Since the monetary distribution is non-degenerate, an agent draws a random trading partner from the distribution $\lambda(dm)$. The second line captures the situation when this agent is a seller, which happens with probability x. The third line describes the case of a non-trader, which happens with probability $1 - 2x$. In any case, the agent receives a money transfer at a rate $\mu - 1$ and the real balance in next period is re-normalized by the money growth rate μ. Since buyers make take-it-or-leave-it offers to sellers, only buyers can receive a

[6] We assume that $V(m)$ is continuous, which is sufficient for $q(m_b, m_s)$ and $d(m_b, m_s)$ to be measurable functions and therefore the integral to be well defined.

positive trade-surplus. Therefore, the value function can be simplified to

$$
\begin{aligned}
V(m;\epsilon_\mu,\lambda) &= x\beta\int_0^\infty \{u[q(m,m_s;\epsilon_\mu,\lambda)]+ \\
&\quad E_{\epsilon'_\mu}\left\{V\left[\frac{m+\mu-1-d(m,m_s;\epsilon_\mu,\lambda)}{\mu};\epsilon'_\mu,\lambda'\right]-V\left[\frac{m+\mu-1}{\mu};\epsilon'_\mu,\lambda'\right]\right\}\Bigg\}\lambda(dm_s) \\
&\quad +\beta E_{\epsilon'_\mu}V\left[\frac{m+\mu-1}{\mu};\epsilon'_\mu,\lambda'\right].
\end{aligned}
$$

The terms of trade $d(m_b,m_s;\epsilon_\mu,\lambda)$ and $q(m_b,m_s;\epsilon_\mu,\lambda)$ solve[7]:

$$
\max_{0\le d\le m_b+\mu-1,q\ge 0} u(q)+E_{\epsilon'_\mu}V\left[\frac{m_b+\mu-1-d}{\mu},\epsilon'_\mu,\lambda'\right]. \tag{1}
$$

subject to

$$
\begin{aligned}
q &= E_{\epsilon'_\mu}\left\{V\left[\frac{m_s+\mu-1+d}{\mu},\epsilon'_\mu,\lambda'\right]-V\left[\frac{m_s+\mu-1}{\mu};\epsilon'_\mu,\lambda'\right]\right\} &(2)\\
\lambda' &= H(\epsilon_\mu,\lambda) &(3)\\
\mu &= G(\lambda)+\epsilon_\mu. &(4)
\end{aligned}
$$

Here, a buyer chooses the terms-of-trade (q,d) to maximize his payoff subject to his liquidity constraint $m_b+\mu-1\ge d$, the participation constraint of the seller (2), the law of motion of the distribution (3), and the policy rule (4).

We now describe the law of motion of the distribution of money holdings H. The law of motion of the money holdings distribution can be described as a Markov process with an associated transition function $\Pi(m,B;\epsilon_\mu,\lambda)$. Intuitively, $\Pi(m,B;\epsilon_\mu,\lambda)$ is the probability that an individual currently with money holdings m will have money holdings in $B\in\mathfrak{B}_{\mathbb{R}_+}$ next period, given the current aggregate state (ϵ_μ,λ). In what follows we will construct such function. Let $T=\{buyer, seller, neither\}$ and define the space $(T,\mathfrak{T})$, where $\mathfrak{T}$ is the σ-algebra. Define the probability measure $\tau:\mathfrak{T}\to[0,1]$, with $\tau(buyer)=\tau(seller)=x$, and $\tau(neither)=1-2x$. Then, $(T,\mathfrak{T},\tau)$ is a measure space. Define an *event* to be a pair $e=(t,m)$, where $t\in T$, $m\in\mathbb{R}_+$. Intuitively, t denotes an agent's trading

[7]Given the assumed continuity of V, it can easily be shown by the *Theorem of the Maximum* that the set of solutions of the maximization problem is non-empty, compact-valued, and upper hemicontinuous. Thus, by the *Measurable Selection Theorem*, it admits a measurable selection. We define $d(m_b,m_s;\epsilon_\mu,\lambda)$ and $q(m_b,m_s;\epsilon_\mu,\lambda)$ to be that selection.

status and m the money holdings of his current trading partner. Let $(E, \mathfrak{E})$ be the space of such *events*, where $E = T \times \mathbb{R}_+$ and $\mathfrak{E} = \mathfrak{T} \times \mathfrak{B}_{\mathbb{R}_+}$. Furthermore, let $\xi : \mathfrak{E} \to [0,1]$ be the product probability measure. Define the mapping $\theta(m,e) : \mathbb{R}_+ \times E \to \mathbb{R}_+$, where

$$\theta(m, e; \epsilon_{mu}, \lambda) = \begin{cases} \frac{m+\mu-1-d(m,\cdot\,;\epsilon_\mu,\lambda)}{\mu}, & \text{if } e = (\mathit{buyer}, \cdot\,); \\ \frac{m+\mu-1+d(\cdot\,,m;\epsilon_\mu,\lambda)}{\mu}, & \text{if } e = (\mathit{seller}, \cdot\,); \\ \frac{m+\mu-1}{\mu}, & \text{otherwise.} \end{cases}$$

We can now define $\Pi : \mathbb{R}_+ \times \mathfrak{B}_{\mathbb{R}_+} \to [0,1]$ to be

$$\Pi(m, B; \epsilon_\mu, \lambda) \equiv \xi(\{e \in E | \theta(m, e; \epsilon_\mu, \lambda) \in B\}).$$

Π is a well defined transition function.[8] Then,

$$\lambda'(B) = H(\epsilon_\mu, \lambda)(B) \equiv \int_0^\infty \Pi(m, B; \epsilon_\mu, \lambda)\ \lambda(dm)\ \ \forall B \in \mathfrak{B}_{\mathbb{R}_+}. \tag{5}$$

A recursive equilibrium is then a law of motion H, a value function V, terms of trade d and q, and monetary policy rule G such that (i) V satisfies (1), (ii) d and q solve the bargaining problem (1), and (iii) H is defined by (5).

4 The Approximate Economy

Approximated Problem

Solving the stochastic heterogeneous agents model of the previous section with a continuum of agents constitutes an impossible technical problem given that the state vector includes the whole distribution of money, which is an infinite-dimensional object. Recently, algorithms have been proposed which summarize the cross-section distribution of agents' characteristics by a small number of moments, therefore making the dimension of the state space tractable, and calculate the law of motion for these state variables using simulation procedures. The best known examples of this approach are those of Den Hann (1997), Rios-Rull (1997), and Krusell and Smith (1998).

[8] By construction, for each m, $\Pi(m, \cdot\,; \epsilon_\mu, \lambda)$ is a probability measure on $(\mathbb{R}_+, \mathfrak{B}_{\mathbb{R}_+})$. Furthermore, given the measurability of $d(\cdot\,, \cdot\,; \epsilon_\mu, \lambda)$, $\Pi(\cdot\,, B; \epsilon_\mu, \lambda)$ is a $\mathfrak{B}_{\mathbb{R}_+}$-measurable function.

An added difficulty of computing equilibria in our model is that it requires us to compute the expected gain of being a buyer - the integral in equation (1) - which requires us to integrate over the money holdings of the sellers. To exactly compute that expectation might require us to keep track of a large number of moments. To overcome this difficulty we follow Algan, Allais and Den Haan (2008) by parameterizing the cross-sectional distribution and using reference moments as in Reiter (2002). Parameterizing the cross-sectional distribution allows us to obtain a numerical solution using standard quadrature and projection techniques. Using reference moments allows us to get a better characterization of the cross-sectional distribution without increasing the number of state variables. Although we will provide a detailed description of our algorithm we direct the reader to Algan, Allais, and Den Haan for further advantages of their algorithm.

We approximate λ by a finite set of moments $s = [s_1, ..., s_N]$. As mentioned above, we follow Algan, Allais, and Den Hann and parameterize the cross-sectional distribution by a polynomial $P(m;\rho)$ of order $\bar{N}$ with coefficients $\rho = [\rho_0, \rho_1, ..., \rho_{\bar{N}}]$ to match not only the moments $s = [s_1, ..., s_N]$ but also a set of extended moments $\tilde{s} = [s_{N+1}, ..., s_{\bar{N}}]$. We will describe in detail the procedure to obtain the extended moments below. By adopting a particular class of approximating polynomials one can reduce this problem to a convex optimization problem.[9] In particular, the polynomial of order $\bar{N}$ is written as:

$$P(m, \rho) = \rho_0 \exp\{\rho_1[m - s_1] + \rho_2[(m - s_1)^2 - s_2] + ... + \rho_{\bar{N}}[(m - s_1)^{\bar{N}} - s_{\bar{N}}]\}.$$

When the density is constructed in this particular way, the coefficients, except for ρ_0, can be found with the following minimization routine:

$$\min_{\rho_1, \rho_2, ..., \rho_{\bar{N}}} \int_0^\infty P(m, \rho)\, dm.$$

The parameter ρ_0 can be determined by the condition that the density integrates to 1, $\int_0^\infty P(m, \rho)\, dm = 1$.

Finally, since we only use a limited set of moments as state variables, $s = [s_1, ..., s_N]$, the transition law only needs to specify how this limited set of moments evolves over time. Thus,

[9] Without the convexity of the problem obtaining convergence can be problematic without finding a good initial condition.

instead of computing H, we now calculate $s' = \Gamma_n(\epsilon_\mu, s; \gamma_n)$ where Γ_n is an n^{th}-order polynomial with coefficients γ_n.

The approximate problem can then be written as:

$$
\begin{aligned}
V(m;\epsilon_\mu,s) \;=\;& x\beta\int_0^\infty \{u[q(m,m_s;\epsilon_\mu,s)]+ && (6)\\
& E_{\epsilon'_\mu}\left\{V\left[\frac{m+\mu-1-d(m,m_s;\epsilon_\mu,s)}{\mu};\epsilon'_\mu,s'\right] \; nonumber && (7)\\
& -V\left[\frac{m+\mu-1}{\mu};\epsilon'_\mu,s'\right]\right\}\Bigg\} P(dm_s;\rho)+\beta E_{\epsilon'_\mu}V\left[\frac{m+\mu-1}{\mu};\epsilon'_\mu,s'\right].
\end{aligned}
$$

$$
\begin{aligned}
s' &= \Gamma_n(\epsilon_\mu, s; \gamma_n).\\
\mu &= \Phi(s)+\epsilon_\mu,
\end{aligned}
$$

where $d(m_b, m_s; \epsilon_\mu, s)$ and $q(m_b, m_s; \epsilon_\mu, s)$ solve:

$$
\max_{0\le d\le m_b+\mu-1,\, q\ge 0} u(q)+E_{\epsilon'_\mu}V\left[\frac{m_b+\mu-1-d}{\mu},\epsilon'_\mu,s'\right].
$$

subject to

$$
\begin{aligned}
q &= E_{\epsilon'_\mu}\left\{V\left[\frac{m_s+\mu-1+d}{\mu},\epsilon'_\mu,s'\right]-V\left[\frac{m_s+\mu-1}{\mu};\epsilon'_\mu,s'\right]\right\}\\
s' &= \Gamma_n\left(\epsilon_\mu, s;\gamma_n\right)\\
\mu &= \Phi\left(s\right)+\epsilon_\mu.
\end{aligned}
$$

In the implementation of the algorithm we only keep track of the second moment $s = s_2$ as a state variable, that is, we set $N = 2$. Note that in this model the first moment is constant at $s_1 = 1$, since the average money holdings as a fraction of the total money stock is 1. For each ϵ_μ we approximate the value function of a grid of money holdings, m, and second moments, s_2. In the m direction we use an uneven grid and a shape-preserving interpolation routine. In the s_2 direction we use an evenly spaced grid and piecewise-linear interpolation. This is done to guarantee that the algorithm preserves the concavity of V with respect to m.

Computing the Reference Moments

In what follows we describe the procedure used to compute the reference moments at a point in the algorithm. For a given guess for the value functions, the law of motion, and the policy function, the idea is to simulate the model to obtain more information regarding the shape of the distribution. The procedure works as follows:

1. Use a random number generator to draw a time series for monetary shocks $\{\epsilon_{\mu t}\}_{t=0}^{T}$.
2. In period 1, the procedure starts with a given set of moments. To start the algorithm, we start with the moments of the stationary equilibrium of a model where the shocks are set to their unconditional mean. We set $\bar{N} = 6$, that is we use 4 additional reference moments. Given the set of moments we use the procedure described above to compute the coefficients of the polynomial used to approximate the density, ρ^1.
3. Given the value function, the law of motion, and the policy rule we compute the decision rules $d(m_b, m_s; \epsilon_\mu, s)$ (on a grid of money holdings and use interpolation). Given d, the monetary transfers, $P(m, \rho^1)$ and the fraction of agents that are buyers and sellers we can then compute the new moments of the distribution using quadrature. Note that in this step we are using the fact that there is a continuum of agents and computing the moments without sampling variation.
4. Given these new moments we use again the procedure to compute the coefficients of the density ρ^2.
5. Repeat the process for the next period until $t = T$.

To ensure that the sample used to obtain information about the cross-sectional distribution has reached (or is at least close to) its ergodic distribution one should disregard an initial set of observations. We set $T = 2000$ and discard 500 observation. Given that using this procedure to compute the reference moments is computationally costly we only update the reference moments every 10 iterations of the algorithm which we describe in more detail in the next section.

5 Numerical Algorithm

We now present a brief description of the algorithm used to compute the recursive equilibrium. The basic strategy of the algorithm is to iterate on the mapping defined by equation (6) starting from an initial guess for $V(m, \epsilon_\mu, s_2)$ and for the law of motion Γ_n. The algorithm works as follows:

- **Compute Reference Moments** - In iteration 1, we begin by computing the set of reference moments using the procedure described in the last section. As mentioned before, we only update the reference moments every 10 iterations given that the procedure is computationally time-costly.

- **Computing the Cross-sectional Distributions** - For each state s_2 on the grid and the associated reference moments previously obtained, we compute the coefficients of the associated densities $P(m, \rho)$.

- **Updating the Value Function** - Given $V(m, \epsilon_\mu, s_2)$, the law of motion Γ_n and the densities $P(m, \rho)$ obtained in the last step, for each quintuplet (m, ϵ_μ, s_2) we use equation (6) to update the V on the state space grid using quadrature methods. For the integration, we define $d(m_b, m_s; \epsilon_\mu, s_2)$ and $q(m_b, m_s; \epsilon_\mu, s_2)$ as functions obtained by solving the bargaining problem when needed. We do not support the decision rules on a grid to avoid introducing approximation errors, which could lead to violations of the participation constraints of the agents, as well as the loss of concavity of the value function.

- **Updating the Law of Motion** - Given $P(m, \rho)$, the monetary policy rule, and the decision rules $d(m_b, m_s; \epsilon_\mu, s_2)$, for each (ϵ_μ, s) on the state space grid, we compute a new set of second moments s_2'. Note that, conditional on the realizations of the aggregate shocks and the current period's cross-sectional distribution, next period's cross sectional distribution is known with certainty. We then use standard projection methods to update the coefficients of Γ_n. Note that this step does not require any kind of simulation procedure and thus avoids introducing cross-sectional sampling variation.

We repeat the steps above until the law of motion has converged.

6 A Numerical Illustration

In what follows, we use the numerical algorithm presented in the last section to find and characterize equilibria of the model. In particular, we characterize the typical features of an equilibrium and illustrate the effects of short-run and long-run monetary policy.

(a) Parameterization

In this numerical example, we set the period length to a quarter ($\beta = 0.99$). We adopt the functional forms from Molico (2006) for the utility and cost functions:

$$\begin{aligned} u(q) &= A * \ln(1+q), \\ c(q) &= \frac{1}{\bar{q}-q} - \frac{1}{\bar{q}}, \end{aligned}$$

with $A = 100, \bar{q} = 1$. We set $\sigma = 0.5$ to minimize the search frictions. The central bank targets a constant growth rate of money: $G(\lambda) = \bar{\mu}$. The annual money growth rate is two percents. The i.i.d. monetary shocks, ϵ_μ, are drawn from the set $\{0.0025, 0, -0.0025\}$. To solve the model, we use the second moment as a state variable and the 3rd to 5th moments as reference moments.

(b) Equilibrium

We find that the law of motion of the endogenous aggregate state is well approximated by a linear function:

$$s_2' = 0.001706 - 0.266 * \epsilon_\mu + 0.986 * s_2, \qquad \text{(with } R^2 = 0.999984)$$

Here, s_2 denotes the second moment in the current period, and s_2' denotes the second moment in the next period. Note that a positive monetary shock decreases the dispersion of money holding (i.e. $\frac{\mathrm{d}s_2'}{\mathrm{d}\epsilon_\mu} = -0.266$), and the distributional effects of monetary shocks are very persistent (i.e. $\frac{\mathrm{d}s_2'}{\mathrm{d}\epsilon_{s_2}} = 0.986$). The idea is that, due to market incompleteness, the equilibrium distribution of monetary holding is non-degenerate. As a result, a lump sum transfer of money will lead to distributional effects, redistributing real money balances from those who are holding excess liquidity to those who are liquidity constrained. This effect tends to reduce the dispersion of money holdings, and thus the coefficient attached to ϵ_μ in the equation above is negative. In the absence of additional monetary shocks, the distribution will converge to the steady state gradually. This is because, in this environment, the only way of insuring was to carry large precautionary money balances, generating a low velocity of money circulation. As a result, it takes time for the money holding to return to its long-run distribution. That explains why the coefficient attached to s_2 is close to one. This feature is very different from the Lagos-Wright type of model in which the distribution of money holding is degenerated and thus monetary shocks have no persistent real effects through the distributional channel.

(c) Impulse Responses: One-time money growth shock

Figure 2 illustrates the impulse responses to an unanticipated positive shock to money growth. As explained above, the shock generates a negative persistent effect on the dispersion of money holding. It also has a positive persistent effect on the aggregate output. The intuition is captured by figure 3 which plots the quantities of goods traded as a function of the money holdings of the buyer (m_b) and the seller (m_s). Note that, for any given m_s, the quantity is concave in the buyer's money holding. So, other things being equal, a drop in dispersion tends to increase the aggregate output. That is, buyers on average tend to buy higher quantities when the real balances are redistributed from the rich to the poor buyers. On the other hand, for any given m_b, the quantity is convex in the seller's money holding. So, other things being equal, a drop in dispersion tends to decrease the aggregate output. That is, sellers on average tend to sell lower quantities when the real balances are redistributed from the rich to the poor buyers. It turns out that, the first effect dominates and the aggregate output increases as the dispersion of the distribution drops. This effect is persistent because the distribution is persistent.

The welfare, measured by average utility in the economy also exhibits a similar positive and persistent response. The intuition is that, since the value function is concave with respect to money holding, a drop in dispersion implies a higher average welfare in the economy. This effect is persistent because the distribution is persistent.

(d) Permanent change of inflation target

Figure 4 illustrates the transition paths of an unanticipated, permanent change of money growth rate from 1% to 2%. Again, the adjustment to the new steady state takes time because the distribution is persistent. Comparing the welfare levels across the new and the old steady state

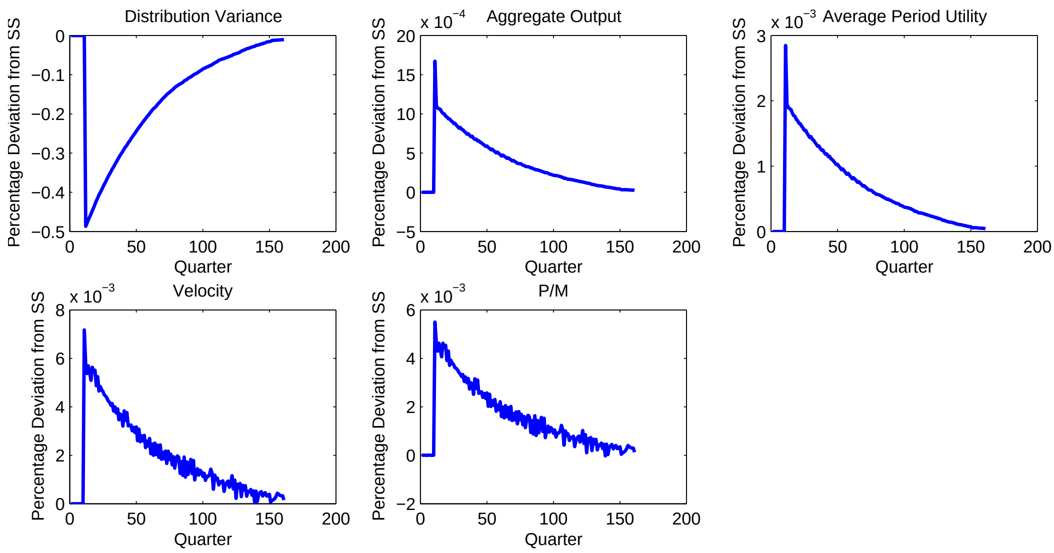

Figure 2: Impulse Response: One-time money growth shock

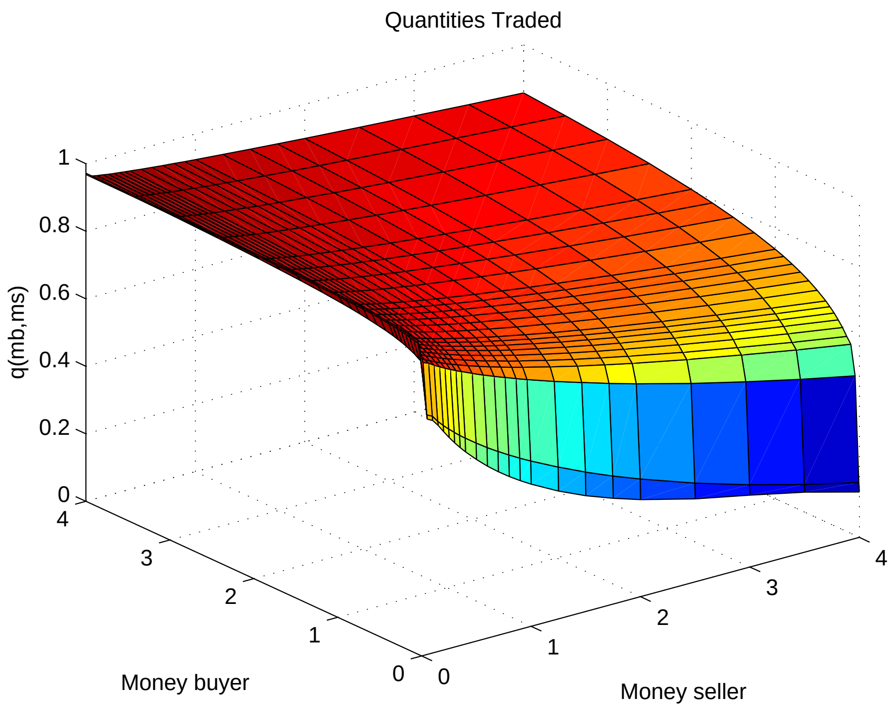

Figure 3: Bargaining Outcome

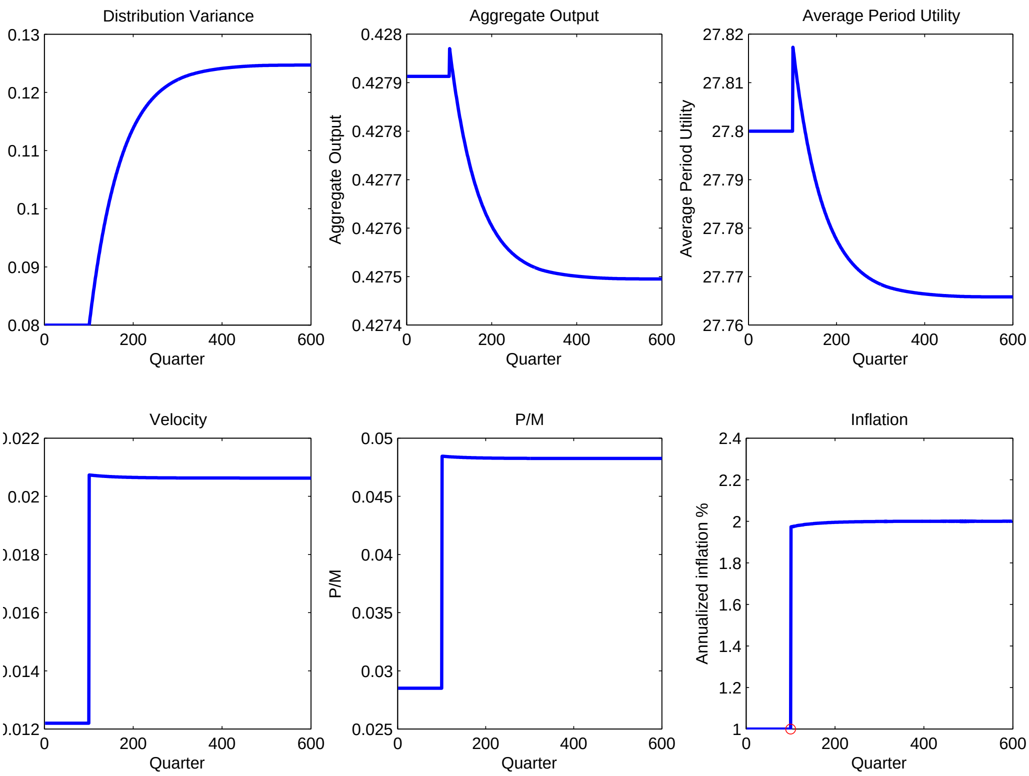

Figure 4: Permanent Change of Inflation: 1% to 2%

equilibria suggests that the welfare cost of moving from 1% to 2% inflation is equivalent to 0.276% of steady state consumption at 1%. Considering also the welfare effects along the transition, the welfare cost goes down to 0.1888%. This is because, while inflation reduces welfare, it takes time for the full effects to realize because the distribution needs time to adjust to the new equilibrium in the presence of trading frictions.

7 Concluding Remarks

This paper studied the internal propagation mechanism of a micro-founded search model of money, and found that monetary policy can have persistent effects on output, prices and welfare, even in the absence of ad-hoc nominal rigidities. In particular, our result highlights the fact that explicitly modelling the micro-foundation of money is important not only for long-run steady state analysis, but also important for understanding the short-run effects of monetary policy.

In this environment, money is the only asset, and agents will tend to hold excess precautionary balances, making it difficult to match the observed velocity of money (Molico, 2006). In reality, other assets (e.g., financial assets, physical capital) also provide such insurance role. The introduction of these assets avoids this excessive holding of money balances, allowing the model to match the empirical money demand. Whether explicitly modeling these assets can have other effects on the welfare costs of inflation is an open question. While this extension is interesting, it is technically challenging since it increases the dimensionality of this already computationally intensive numerical problem, and is left for future research.

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