Optimality in an Adverse Selection Insurance Economy with Private Trading

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Abstract

An externality is created in an adverse selection insurance economy because of the interaction between private information and the possibility of private trading. The consumption possibility set of an agent depends on the collective decisions of all agents through the incentive compatibility constraints, which include the prices and opportunities created by private trading. When agents have private information about the distribution of their idiosyncratic endowment shocks, decentralizing an incentive-efficient allocation generally requires separation of markets by type, exclusivity of contracts and strict prohibition of private trading. If these restrictions are dropped, agents will trade to eliminate arbitrage profits, resulting in private trading prices that are typically not actuarially fair for any type of agent and imposing limits on risk sharing. Agents will choose to under or over insure even though full consumption insurance is in their budget set, and essentially trade probability distributions, as in the reinsurance market. When agents trade contingent claims at common prices, the competitive equilibria may not be private-trading constrained efficient (third best) because of inefficient cross-subsidization across different types of agents. Imposing a simple price ceiling on the price of insurance will restore efficiency.

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Private information creates frictions in trading that are important in understanding the structure of financial markets and risk-sharing more broadly. When private trading opportunities interact with private information, an externality is created in an adverse selection insurance economy. The individual incentive compatibility constraints create an externality in that the net trades of one set of agents must be incentive compatibile with respect to the net trades of other agents. When private trading opportunities are added, the incentive compatibility constraints are modified to incorporate how private trading opportunities, including prices, alter the incentives to reveal information. Satisfying the incentive compatibility constraints with private trading prevents implementing incentive-efficient allocations because the required restrictions are not satisfied, such as separation of markets by type, exclusivity of contracts and strict prohibition of private trading. Private-trading constrained efficient allocations can be defined within the context of the model. Competitive equilibria are not generally private trading constrained efficient because of inefficient cross-subsidization across different types of agents. Imposing a simple price ceiling on the price of insurance will restore efficiency.

When private trading cannot be prevented, agents reveal information about their type based on the value of an allocation in the private trading market. An agent is better off announcing he is the type providing him the highest market value of an allocation in private trading, with the result agents face the identical budget set in private markets. This is the basis for a convenient reformulation of the social planner's problem, just as in Farhi, Golosov and Tsyvinski [10]. In the original problem, the social planner chooses consumption allocations satisfying the feasibility constraint and the modified incentive compatibility constraint based on the indirect utility an agent would achieve through trading in private markets at equilibrium prices. In the reformulated problem, the social planner picks the market value of the allocation and equilibrium price. The conditions for existence and uniqueness of competitive equilibria are established and the two social planning problems are shown to be equivalent under certain conditions.

The existence of competitive equilibria in adverse selection insurance economies is often problematic. An incentive-efficient allocation, which is constrained efficient (second best), is difficult to decentralize, in part because the set of incentive compatible allocations is not convex, creating a consumption externality; see Prescott and Townsend [14]. As discussed by Bisin and Gottardi [6], a competitive equilibrium with exclusivity, separation of markets and no private trading may not be incentive efficient. In these settings, a contract is a bundle of contingent claims with restricted quantities and a zero-profit condition resulting in contracts that are actuarially fair with no cross-subsidization. When private trading cannot be prevented, agents have an incentive to unbundle the contingent claims in an insurance contract to eliminate arbitrage profits and to improve risk-sharing. As Bisin and Gottardi [5]-[6] and Rustichini and Siconolfi [17] have shown, the problems of existence of competitive equilibria are not mitigated by allowing price-taking agents to trade standardized contingent-claims contracts among themselves. One contribution of this paper is to characterize restrictions on endowments such that an equilibrium exists, although it may not be unique.

The non-exclusivity of contracts in adverse selection insurance models has been studied in many papers, recently by Ales and Maziero [1] and Attar, Mariotti, and Salanie [3]. Nonexclusivity in these settings implies agents can contract with more than one principal and the terms of the contract are private information. Ales and Maziero, for example, address non-exclusivity assuming insurance is sold through an intermediary, there is free entry, and agents can sign multiple contracts with intermediaries. They determine conditions for a separating equilibrium. Attar, Mariotti and Salanie [3] provide necessary and sufficient conditions for the existence of a pure strategy equilibrium. While there are important differences between the papers, they share common features such as any contract traded in equilibrium yields zero profits, so there are no cross subsidizations across types and trading takes place among agents specializing in either buying or selling of insurance. I assume agents directly trade among themselves through standardized contracts and cross subsidization across types is a feature of an equilibrium.

The basic model is described in Section 1 and the incentive-efficient allocation is described in Section 2, and the implications of private trading and non-exclusivity of contracts are discussed. The problem of constrained efficiency with private trading is stated in Section 3 along with a convenient reformulation of the problem. The existence of equilibria is problematic and this issue is addressed in in Section 3.2. The solution to the constrained efficient problem with private trading is derived in Section 4. Implementation of the optimal allocation is discussed in Section 4.1. The market structure and elimination of arbitrage profit opportunities is discussed in the Appendix.

1 Basic Model

The basic model is the adverse selection insurance economy of Rothschild and Stiglitz [15], Prescott and Townsend [14], Bisin and Gottardi [5], and Labadie [13].

This is a single-period pure endowment economy with a consumption good that is tradeable and divisible. There is a continuum of agents indexed over the unit interval and two types of agents, a and b. A fraction f_a of agents are type a and $f_b = 1 - f_a$ are type b. An agent's type is private information.

The endowment θ is a discrete random variable taking two values $0 \leq \theta_1 < \theta_2$, so θ_1 is the "bad" state and θ_2 is the "good" state. The random variable θ is independently distributed across agents. For a type η agent, the probability of drawing θ_i is $g_{\eta i}$, $\eta \in \{a, b\}$ and $i \in \{1, 2\}$. Let $g_{a2} > g_{b2}$ so type a agents are "low" risk and type b are "high" risk. Denote

$$R_\eta \equiv \frac{g_{\eta 1}}{g_{\eta 2}} \qquad \text{for} \qquad \eta \in \{a, b\},$$

as a measure of type risk, where $R_b > R_a$. The realization of θ is public information.

Let

$$\bar{\theta}_\eta = g_{\eta 1}\theta_1 + g_{\eta 2}\theta_2 \qquad \eta = a, b$$

denote the expected endowment for type η, where $\bar{\theta}_a > \bar{\theta}_b$, and let

$$\bar{\theta} = f_a\bar{\theta}_a + f_b\bar{\theta}_b \tag{1}$$

denote average endowment. Define the (unconditional) probability of realizing θ_1 as

$$p \equiv \sum_\eta f_\eta g_{\eta 1}$$

and let $\bar{R} \equiv \frac{p}{1-p}$.

The following assumption is standard and states the probability of being in any endowment state is strictly positive for either type of agent.

Assumption 1 $g_{\eta i} > 0$, $\eta = a, b$, $i = 1, 2$.

A type-η agent has preferences

$$\sum_i g_{\eta i} U(c_i). \tag{2}$$

Assumption 2 *The function U is twice continuously differentiable, strictly increasing and strictly concave. As $c \to 0$, $U'(c) \to \infty$ (the Inada condition).*

The expectation in (2) uses an agent's conditional probability of realizing θ. Since there is no aggregate uncertainty, the only risk an agent faces is his idiosyncratic endowment risk.

A consumption allocation for a type η agent is $c(\eta) = (c_1(\eta), c_2(\eta))$, where $c(\eta) \in R^2_+$. A pair of consumption allocations $(c(a), c(b)) \in R^4_+$ is feasible in the aggregate if the economy-wide resource constraint holds. Since there is a continuum of agents, the Law of Large Numbers implies the set of consumption allocations feasible in the aggregate is

$$F \equiv \left\{ (c(a), c(b)) \in R^4_+ \mid \bar{\theta} \geq \sum_{\eta} \sum_{i} f_\eta g_{\eta i} c_i(\eta) \right\}. \tag{3}$$

Definition 1: The pair of consumption allocations $(c(a), c(b)) \in F$ is *incentive compatible* if

$$\sum_i g_{\eta i} U(c_i(\eta)) \geq \sum_i g_{\eta i} U(c_i(h)), \qquad h \neq \eta, \qquad h, \eta = a, b. \tag{4}$$

Let $I \subset F$ denote the set of feasible consumption allocations that are incentive compatible. The set I is generally not convex and, as discussed by Prescott and Townsend [14], this non-convexity makes the application of standard general equilibrium analysis problematic. There is a consumption externality created by the constraint (4) because the net trades of an agent in one market must be individually incentive compatible with respect to the net trades of an agent in the other market.

2 Incentive Efficiency and Private Trading

When private trading among agents is prohibited and enforced, the social planner determines the state-contingent consumption of all agents and is able to implement an incentive-efficient allocation.

Definition 2: The consumption allocation $(c(a), c(b)) \in I$ is *incentive efficient* if there is no other pair $\hat{c} \in I$ such that

$$\sum_i g_{\eta i} U(\hat{c}_i(\eta)) \geq \sum_i g_{\eta i} U(c_i(\eta)), \qquad \eta = a, b, \tag{5}$$

with strict inequality for at least one type.

Incentive-efficient allocations have been studied extensively by Prescott and Townsend, among others. As they have proven, there are three types of incentive-efficient allocations for this economy: (i)-(ii) full insurance for type η and partial insurance for type h, $\eta \neq h$ and $\eta, h \in \{a, b\}$ (separating); (iii) full insurance for both $c_i(\eta) = \bar{\theta}$ (pooling).

To decentralize an incentive-efficient allocation requires separation of markets by type, prohibition of private trading, and exclusivity of contracts. Agents can enter only one market - A or B (separation), one contract (exclusivity), and must consume the quantities specified in the contract (no private trading). Agents are offered a menu of contracts by a principal, typically an insurance company, and agents can enter into one contract. Any contract traded in equilibrium yields zero expected profits because of competition among principals. All agents are price takers and equilibrium prices are actuarially fair. As a result, there is no cross-subsidization across types. A pooling allocation that is incentive efficient cannot be decentralized. While not all competitive equilibria are incentive efficient, implying the first welfare theorem may not hold, a constrained version of the second welfare theorem does hold: Any incentive-efficient separating allocation can be decentralized as a competitive equilibrium with transfers. A detailed description is contained in Bisin and Gottardi [6].

Suppose the social planner chooses a separating allocation $\{\bar{c}(a), \bar{c}(b)\}$ that is incentive-efficient, but private trading among agents cannot be prevented. After an agent self-selects into a market and purchases an insurance contract, but before observing the realization of his endowment, agents can enter into risk-sharing arrangements with other agents. For simplicity, focus on the separating allocation in which the high-risk agent self-selects into market B and has full consumption insurance equal to $\bar{c}$. High-risk agents have no incentive to engage in further trading with other agents in market B.

The low-risk agent self-selects the contract $\bar{c}(a) \equiv (\bar{c}_{a1}, \bar{c}_{a2})$ providing less than full insurance. Low-risk agents would like to "unbundle" the contract by separating its state-contingent components and then trading among themselves. Suppose a private market opens in which claims can be traded at prices $\hat{q}_{ai}$. The agent chooses $x_a = (x_1, x_2)$ to maximize his objective function subject to his private-market budget constraint

$$\hat{q}_{a1}\bar{c}_{a1} + \hat{q}_{a2}\bar{c}_{a2} \geq \hat{q}_{a1}x_1 + \hat{q}_{a2}x_2, \tag{6}$$

where the left side is the private market value of the contract selected by a type a agent. **If** only type a agents enter the original market A and the subsequent private market, then the private market-clearing prices are $\hat{q}_{ai} = g_{ai}$ and the low-risk agent chooses constant consumption $\hat{c}_a \equiv g_{a1}\bar{c}_{a1} + g_{a2}\bar{c}_{a2}$. At this point, there are no further gains from trade.

The result of private trading is low-risk agents are better off while high-risk agents are no worse off. But this creates the following difficulty: If high-risk agents know low-risk agents intend to trade in private markets, then a high-risk agent may have an incentive to misrepresent his type so he too can participate in the private market. If $\hat{c}_a > \bar{c}_b$, then a type b agent will misrepresent his type to enter market A. But if all agents self-select into market A, then $\hat{c}_a$ is no longer feasible at the prices (g_{a1}, g_{a2}) and the self-selection into separate markets breaks down. The difficulty introduced by private trading is the new allocation $\hat{c}_a$ does not satisfy the original incentive compatibility constraints (4). The existence of subsequent trading opportunities generally changes the incentives for revealing information.[2]

3 Private Trading

As discussed in the previous section, the existence of private-trading arrangements implies the agents can choose to consume a convex combination of the allocation offered by a social planner by engaging in private trading. This process of unbundling the state-contingent components of any allocation is the key property of private trading and motivates the market structure described in this section.

Suppose agents can trade contingent claims and are price takers, as in Bisin and Gottardi [5]. Let $q_{\eta i}$ denote the price of a contingent claim in market $\eta \in \{A, B\}$ for state $i \in \{1, 2\}$. For simplicity, normalize prices so

$$1 = q_{\eta 1} + q_{\eta 2} \qquad \eta \in \{a, b\}$$

and assume $q_{\eta i} > 0$. Markets are said to be separate if $q_{\eta i} \neq q_{hi}$ for $\eta, h \in \{a, b\}$, $h \neq \eta$, $i \in \{1, 2\}$ and there is trading in both markets. If the relative price differs across markets, then an arbitrage opportunity exists because contracts are not exclusive, and elimination of arbitrage profit opportunities will result in the equalization of prices across markets, so $q_{ai} = q_{bi}$, if an equilibrium exists. The elimination of arbitrage profits and the emerging market structure are discussed in Part B of the Appendix.

If the social planner chooses the pooling allocation, so $c_{\eta i} = \bar{\theta}$ for all η and i, then there is no incentive to enter into private-trading arrangements because agents are identical across states and types. Whether the

[2]The idea subsequent trading opportunities can change the information revealed by an agent is studied by Krasa [12], who examines private information exchange economies with allocations that cannot be "improved," in the agent would not wish to deviate by revealing further information or by the retrading of goods.

social planner chooses the pooling allocation will depend on the Pareto weights. In the discussion below, I assume the social planner chooses a separating allocation.

3.1 Competitive Equilibrium with Private Trading

Agents are offered a separating allocation in the form of a menu of contracts $\{c(a), c(b)\} \in I$. An agent makes an announcement of his type and it is assumed all agents of the same type make the same announcement (symmetry). For convenience, normalize prices so $1 = q_1 + q_2$ and assume $q_i > 0$. Denote $q = q_1$ so $q_2 = 1 - q$ and let $q \in Q \equiv (0, 1)$. An agent treats as fixed the menu of contracts $\{c(a), c(b)\}$ and the price q in the private market.

An agent chooses his optimal reporting strategy $h \in \{a, b\}$, determining his allocation $c(h) = (c_1(h), c_2(h))$. His after-trade consumption (x_1, x_2) may differ from his allocation $c(h)$ if he chooses to engage in private trading. An agent reporting type h faces a budget set

$$\hat{B}(c(h), q) \equiv \{x \in R^2_+ \mid qx_1 + (1-q)x_2 \le qc_1(h) + (1-q)c_2(h)\} \tag{7}$$

in private markets. Given the menu of contracts $\{c(a), c(b)\}$ and relative price $q \in Q$, a type η agent solves

$$\hat{V}(\{c(a), c(b)\}, q; \eta) = \max_{\{h, x_1, x_2\}} \sum_i g_{\eta i} U(x_i) \tag{8}$$

subject to $h \in \{a, b\}$ and

$$(x_1, x_2) \in \hat{B}(c(h), q). \tag{9}$$

Denote the solution by $\hat{h}(\{c(a), c(b)\}, q; \eta)$ and $\hat{x}(\{c(a), c(b)\}, q; \eta) = \{\hat{x}_1(\{c(a), c(b)\}, q; \eta), \hat{x}_2(\{c(a), c(b)\}, q; \eta)\}$ for $\eta \in \{a, b\}$.

Definition 4: Given a separating allocation $\{c(a), c(b)\} \in I$, an *equilibrium in private markets* consists of a relative price $q \in Q$ and, for each $\eta \in \{a, b\}$, consumption allocations and a reported type such that

(i). $\hat{x}(\{c(a), c(b)\}, q; \eta)$ and $\hat{h}(\{c(a), c(b)\}, q; \eta)$ for $\eta \in \{a, b\}$ solve (8) subject to $h \in \{a, b\}$ and the budget constraint (9).

(ii). Markets clear:

$$\sum_\eta f_\eta \sum_i g_{\eta i} \hat{x}_i(\{c(a), c(b)\}, q; \eta) \le \sum_\eta f_\eta \sum_i g_{\eta i} c_i(\hat{h}(\{c(a), c(b)\}, q; \eta)). \tag{10}$$

The equilibrium in private markets has the property an agent's allocation is a function of his announced type. A competitive equilibrium with private trading is an equilibrium with private markets such that the announcement-based allocation $\{c(\hat{h}(\{c(a), c(b)\}, q; a)), c(\hat{h}(\{c(a), c(b)\}, q; b))\}$ is feasible.

Definition 5: Given a separating allocation $\{c(a), c(b)\} \in F$, a *competitive equilibrium with private trading* is a relative price $q \in Q$ and an allocation $(\hat{x}(\{c(a), c(b)\}, q; a), \hat{x}(\{c(a), c(b)\}, q; b))$ such that

(i) The relative price q, allocation $(\hat{x}(\{c(a), c(b)\}, q; a), \hat{x}(\{c(a), c(b)\}, q; b))$, and announcement $\hat{h}(\{c(a), c(b)\}, q; \eta)$, for $\eta = \{a, b\}$, are an equilibrium in private markets (Definition 3);

(ii) $\eta = \hat{h}(\{c(a), c(b)\}, q; \eta)$ for $\eta \in \{a, b\}$ (truth-telling);

(iii) Feasibility: The announcement-based allocation $\{c(\hat{h}(\{c(a), c(b)\}, q; a)), c(\hat{h}(\{c(a), c(b)\}, q; b))\} \in F$.

The existence of a competitive equilibrium is discussed in Section (3.2). The constrained-efficient social planning problem with private trading is described next.

3.2 Constrained Efficiency with Private Trading

The social planner offers a contract maximizing the Pareto-weighted expected utilities. Let Ψ_η denote a Pareto weight for a type η agent, with $\Psi_\eta > 0$ and $1 = \Psi_a + \Psi_b$. The constrained-efficient allocation with private trading is the solution to

$$\max_{\{c(a),c(b)\}} \sum_{\eta} \Psi_\eta \sum_i g_{\eta i} U(c_i(\eta)) \tag{11}$$

subject to

$$\bar{\theta} \geq \sum_{\eta} f_\eta \sum_i g_{\eta i} c_i(\eta), \tag{12}$$

$$\sum_i g_{ai} U(c_i(a)) \geq \hat{V}(\{c(a), c(b)\}, q; a), \tag{13}$$

$$\sum_i g_{bi} U(c_i(b)) \geq \hat{V}(\{c(a), c(b)\}, q; b). \tag{14}$$

The first constraint (12) is feasibility of the allocation. The second and third constraints, (13) and (14), replace the individual incentive compatibility constraints (4), where the right side is the indirect utility function of an agent trading in private markets. The consumption allocation is required to provide expected utility at least as high as the type η agent can achieve by reporting he is type $\hat{h}(\{c(a), c(b)\}, q; \eta)$ and then trading in private markets at relative price q.

The allocation has three key properties: (i) each agent reports his type truthfully; (ii) each agent has a zero net trade position in private markets in the sense $c(\eta)$ is the solution to (8); (iii) The individual incentive compatibility conditions are satisfied for each type and hold as strict inequalities.[3] This problem can be conveniently reformulated.

An Alternate Specification

Farhi, Golosov and Tsyvinski (FGT) study a Diamond-Dybvig model with retrading and prove an agent will base his announcement of type on the private-market value of an allocation. A similar argument can be applied in this adverse selection setting to reformulate the problem described in (11)–(14). In this alternate specification, the social planner chooses the market value of an allocation instead of the individual components to solve the constrained efficient problem with private trading.

For a given contract $\{c(a), c(b)\}$ that is separating and a price $q \in Q$, an agent's allocation will depend on his announced type h. If there is an h such that

$$qc_1(h) + (1-q)c_2(h) > qc_1(\eta) + (1-q)c_2(\eta) \tag{15}$$

for $\eta, h \in \{a, b\}$ and $\eta \neq h$, then each agent will announce he is a type h, regardless of his true type. This follows because the budget set $\hat{B}(c(h), q)$ is increasing in $c(h)$ and the indirect utility function is strictly increasing in its first argument

$$\hat{V}(\{c(a), c(b)\}, q; \eta) = \hat{V}(\max[qc_1(a) + (1-q)c_2(a), qc_1(b) + (1-q)c_2(b))], q; \eta).$$

Hence, if (15) holds for $h, \eta \in \{a, b\}$ and $h \neq \eta$, then

$$h(\{c(a), c(b)\}, q) \equiv \hat{h}(\{c(a), c(b)\}, q; a) = \hat{h}(\{c(a), c(b)\}, q; b)$$

and the function h can be used to define the certain endowment $w \in [0, \bar{\theta}]$

$$w \equiv qc_1(h(\{c(a), c(b)\}, q)) + (1-q)c_2(h(\{c(a), c(b)\}, q)).$$

[3]As will be shown below, each agent will announce the type with the highest market value in private markets and will, therefore, face the same budget set. From the definition of $\hat{V}$, the allocation $\hat{x}(\{c(a), c(b)\}, q; a), \hat{x}(\{c(a), c(b)\}, q; b)$ will satisfy the individual incentive compatibility constraints (4).

Define the budget constraint for an agent facing relative price q in private markets by

$$B(w,q) \equiv \{x \in R_+^2 \mid w \geq qx_1 + (1-q)x_2\}. \tag{16}$$

The type η agent solves

$$V(w,q;\eta) = \max_{\{x \in B(w,q)\}} \sum_i g_{\eta i} U(x_i). \tag{17}$$

Denote the solution by $\xi_\eta(w,q) \equiv (\xi_{\eta 1}(w,q), \xi_{\eta 2}(w,q))$.

Definition 5: A *competitive equilibrium in private markets* with market endowment $w \in [0,\bar{\theta}]$ consists of a relative price $q \in Q$ and consumption allocations $\{\xi_a(w,q), \xi_b(w,q)\}$ such that

(i). The demand functions $\xi_\eta(w,q)$ solve (17), for $\eta \in \{a,b\}$.

(ii). Markets clear:

$$\bar{\theta} \geq \sum_\eta f_\eta \sum_i g_{\eta i} \xi_{\eta i}(w,q). \tag{18}$$

In the modified problem, the social planner maximizes the Pareto-weighted indirect utility by picking $w \in [0,\bar{\theta}]$ and relative price $q \in Q$ to solve

$$\max_{\{w \in \Theta, q \in Q\}} \sum_\eta \Psi_\eta V(w,q;\eta) \tag{19}$$

subject to

$$\bar{\theta} \geq \sum_\eta f_\eta \sum_i g_{\eta i} \xi_{\eta i}(w,q). \tag{20}$$

Let $(w^\star, q^\star)$ denote the solution; the existence of a solution is discussed in Section (3.2).

The two approaches to determining the constrained-efficient allocation with private trading - the original problem (11)–(14) and the modified problem (19)–(20) - are illustrated in Figure (1). The common endowment point in autarky is $E = (\theta_1, \theta_2)$. The average feasibility constraint is the line through the point $(\bar{\theta}, \bar{\theta})$ with slope $-\bar{R}$, horizontal intercept $\left(\frac{\bar{\theta}}{p}, 0\right)$ and vertical intercept $\left(0, \frac{\bar{\theta}}{1-p}\right)$. The budget constraint for an agent is the line through the point (w,w) with slope $\left(-\frac{q}{1-q}\right)$, horizontal intercept $\left(\frac{w}{q}, 0\right)$, and vertical intercept $(0, \frac{w}{1-q})$.

In the first problem (11)–(14), the social planner chooses consumption allocations $\{c(a), c(b)\}$, located on the common budget set in Figure 1. The expected consumption of a type η agent is a point on the 45-degree line equaling $\bar{c}_\eta \equiv g_{\eta 1} c_1(\eta) + g_{\eta 2} c_2(\eta)$, for $\eta \in \{a,b\}$. In the second problem (19)–(20), the social planner chooses $w \in \Theta$ and $q \in Q$. Each agent, regardless of type, engages in trading subject to the budget constraint and a type η agent will pick consumption $c(\eta) = \xi_\eta(w, q)$. The budget constraint intersects the average resource constraint at the point S.

The equilibrium allocation $\{\xi_a(w, q), \xi_b(w, q)\}$ corresponds to a pair of contingent claims contracts that are typically not actuarially fair. The rate of exchange between state 1 and state 2 consumption, the slope of the budget constraint $-\frac{q}{1-q}$, can be thought of as measuring the cost of "insurance" in state 1. Fair insurance means the amount expected to be paid out by an agent equals the amount collected from him in premiums. If $l_\eta(w, q) = \xi_{\eta 1}(w, q) - w$ is the amount of consumption shifted to state 1 (insurance), then the payments out of state 2 consumption are $\frac{q}{1-q} l_\eta(w, q)$ and his expected claim is $R_\eta l_\eta(w, q)$. The distortion per unit of insurance for a type η agent is

$$\tau_\eta(q) \equiv g_{\eta 1} - \frac{g_{\eta 2} q}{1-q} = g_{\eta 2}\left[R_\eta - \frac{q}{1-q}\right].$$

The expected subsidy received by a type η agent, equal to the amount of insurance times the claims minus payments

$$S_\eta(w, q) \equiv l_\eta(w, q)\tau_\eta(q),$$

is generally nonzero.

This subsidy can be used to explain the difference between average endowment $\bar{\theta}$ and the market endowment w. Rewrite the resource constraint with the demand functions using the definition of l_η, where $\xi_{\eta 2}(w, q) = w - \frac{q}{1-q} l_\eta(w, q)$, to show

$$\bar{\theta} = \sum_\eta f_\eta \left[g_{\eta 1}\left(w + l_\eta(w, q)\right) + g_{\eta 2}\left(w - \frac{q}{1-q} l_\eta(w, q)\right)\right]$$

or

$$\bar{\theta} - w = \sum_\eta f_\eta g_{\eta 2} S_\eta(w, q) = (\theta_2 - \theta_1)(q - p).$$

The consumption insurance offered at common prices creates subsidies across the agent types which do not net to zero, so feasibility requires agents receive market endowment $w = \Lambda(q)$ such that $w < \bar{\theta}$ for all $q \in Q$.

Before solving the social planner's problem, it is critical to establish the conditions under which a competitive equilibrium exists in private markets.

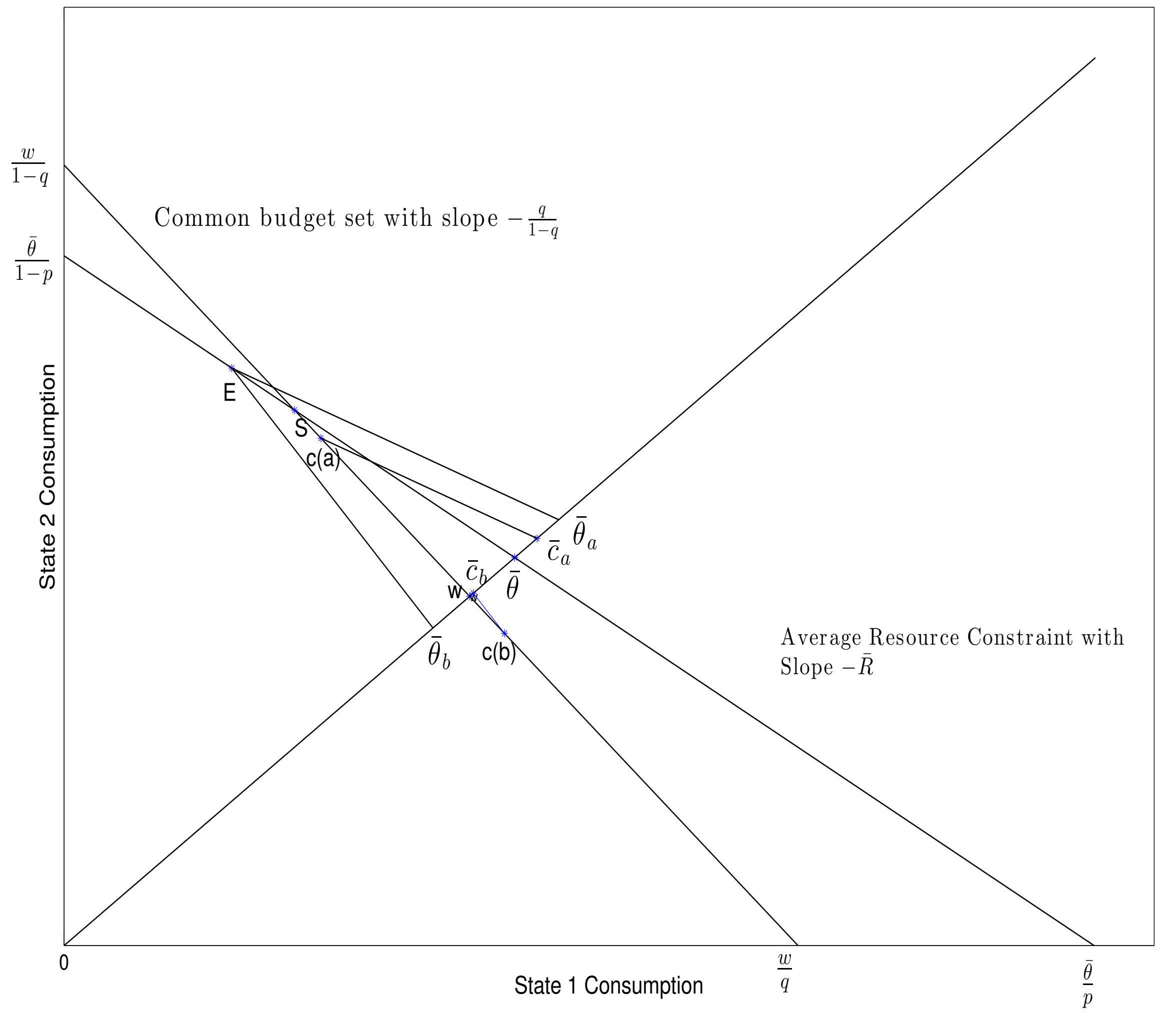

Figure 1: *Budget set and equilibrium with private trading. Point E is the autarky endowment point. The allocation $\{c(a), c(b)\}$ is the solution to the problem (11)–(14) and lies on the common budget constraint. In the second approach, agents have endowment w, face relative price $\frac{q}{1-q}$ and engage in trade to consume $\{c(a), c(b)\}$.*

3.3 Equilibria with Private Trading

As discussed in Bisin and Gottardi [5]–[6], Prescott and Townsend [14], and Rustichini and Siconolfi [17], among others, the existence of a competitive equilibrium can be problematic in these types of models. For a given $w \in [0, \bar{\theta}]$, or more generally given a state-contingent, feasible and common endowment (θ_1, θ_2), an equilibrium may not exist. It is important to emphasize, however, the social planner chooses (w, q) jointly and it is straightforward to show, for any $q \in Q$, there exists a $w \in [0, \theta]$ such that an equilibrium exists. Specifically, for any $q \in Q$, there exists a level of endowment $w \in [0, \theta]$ such that $(\xi_a(w, q), \xi_b(w, q))$ solve (17) and the allocation is feasible, i.e. satisfies (18).

To develop the argument, observe the solution $\xi_\eta(w, q)$ to (17) satisfies the first-order condition

$$R_\eta \frac{U'(\xi_{\eta 1}(w, q))}{U'(\xi_{\eta 2}(w, q))} = \left[\frac{q}{1-q}\right]. \tag{21}$$

Since U is strictly concave and satisfies the Inada conditions and the budget set is convex, it follows the solution $\xi_\eta(w, q)$, $\eta \in \{a, b\}$ exists and is unique. Moreover $\xi_\eta(w, q)$ is continuous, continuously differentiable, and strictly increasing in w. Also $\xi_{\eta 1}(w, q)$ is strictly decreasing in q. Since both agents have identical budget sets $B(w, q)$, the demand functions $\xi_\eta(w, q)$ are incentive compatible for any (w, q). It follows from monotonicity the demand functions have the following properties:

(i) Since $R_b > R_a$, high-risk agents (type b) purchase more consumption in state 1 than low-risk agents (type a),

$$\xi_{b1}(w, q) > \xi_{a1}(w, q) \qquad \text{for all} \qquad q \in Q \qquad \text{and} \qquad w \in [0, \theta].$$

(ii) If the price is actuarially fair $q = g_{\eta 1}$ for a type η agent, then $\xi_{\eta i}(w, g_{\eta 1}) = w$. Agents take on consumption risk if $\frac{q}{1-q} \neq R_\eta$, specifically

$$\xi_{\eta 1}(w, q) \lesseqgtr w \qquad \text{as} \qquad \frac{q}{1-q} \gtreqless R_\eta.$$

(iii) Under the assumptions on the utility function, it follows

$$\lim_{q \to 0} \xi_{\eta 1}(w, q) = +\infty \qquad \text{and} \qquad \lim_{q \to 1} \xi_{\eta 1}(w, q) = 0. \tag{22}$$

(iv) The properties of the demand functions and the assumptions on the utility function imply the indirect utility function $V(w, q; \eta)$ is quasi-convex, continuously differentiable, and strictly increasing in w.

The response of V to changes in q depends on whether agents take on consumption risk.[4]

The feasibility condition after substituting in the demand functions is

$$\bar{\theta} = \sum_{\eta} f_{\eta} \sum_{i} g_{\eta i} \xi_{\eta i}(w, q). \tag{23}$$

The next theorem shows for any $q \in Q$, there exists a unique $w \in [0, \bar{\theta}]$ solving (23) and this equilibrium is separating.

Theorem 1 *To every $q \in Q$, there corresponds a unique $w \in [0, \bar{\theta}]$ solving (23). Define this value by $\Lambda(q)$, where Λ is continuously differentiable in q and satisfies*

$$\bar{\theta} = \sum_{\eta} f_{\eta} \sum_{i} g_{\eta i} \xi_{\eta i}(\Lambda(q), q). \tag{24}$$

Moreover,

$$0 < \Lambda(q) < \bar{\theta} \qquad \textit{for all} \qquad q \in Q.$$

Proof.

This is an application of the implicit function theorem [cf Rudin pp 223]. Since $\xi_{\eta i}(w, q)$ is continuously differentiable in w and q, and strictly increasing in w, the properties of the function Λ are established. The property $0 < \Lambda(q)$ is obvious because $\xi_{\eta}(w, q)$ is strictly increasing in w and $q \in Q$.

To demonstrate $\Lambda(q) < \bar{\theta}$, suppose to the contrary $\Lambda(q) \geq \bar{\theta}$. If $\Lambda(q) > \bar{\theta}$, then the consumption allocation (w, w) lies on the budget set of both types of agents, yet the allocation isn't feasible. If $\Lambda(q) = \bar{\theta}$, then the allocation is clearly feasible, but there is no (common) equilibrium price at which agents would consume $\bar{\theta}$. To see this, substitute the budget constraint into (23) and rewrite,

$$\bar{\theta} - \frac{(1-p)w}{1-q} = \sum f_{\eta} g_{\eta 2} \xi_{\eta 1}(w, q) \left[R_{\eta} - \frac{q}{1-q} \right].$$

When $w = \bar{\theta}$, this expression becomes

$$\bar{\theta} \left[\frac{p-q}{1-q} \right] = \sum f_{\eta} g_{\eta 2} \xi_{\eta 1}(\bar{\theta}, q) \left[R_{\eta} - \frac{q}{1-q} \right].$$

[4]The statement follows because

$$\frac{\partial V_{\eta}(w, q)}{\partial q} = \frac{g_{\eta 2} U'(\xi_{\eta 2}(w, q))}{1-q} [\xi_{\eta 2}(w, q) - \xi_{\eta 1}(w, q)].$$

The sign is determined by the sign of $[\xi_{\eta 2}(w, q) - \xi_{\eta 1}(w, q)]$, which is positive if $q < g_{\eta 1}$, 0 if $q = g_{\eta 1}$, and negative if $q > g_{\eta 1}$.

Consider first $\frac{q}{1-q} \in (0, R_a]$. Since $\frac{q}{1-q} \leq R_a < R_b$ it follows $\bar{\theta} \leq \xi_{a1}(\bar{\theta}, q) < \xi_{b1}(\bar{\theta}, q)$ so

$$\bar{\theta}\left[\frac{p-q}{1-q}\right] > \sum f_\eta g_{\eta 2}\bar{\theta}\left[R_\eta - \frac{q}{1-q}\right] = \bar{\theta}\left[\frac{p-q}{1-q}\right],$$

which is a contradiction. If $\frac{q}{1-q} \in (R_a, R_b]$, then $\xi_{a1}(\bar{\theta}, q) < \bar{\theta}$ and $\xi_{b1}(\bar{\theta}, q) \geq \bar{\theta}$ so

$$\bar{\theta}\left[\frac{p-q}{1-q}\right] > \sum f_\eta g_{\eta 2}\bar{\theta}\left[R_\eta - \frac{q}{1-q}\right] = \bar{\theta}\left[\frac{p-q}{1-q}\right],$$

which is a contradiction. Finally, for $\frac{q}{1-q} \in (R_b, \infty)$, observe $\xi_{\eta 1}(\bar{\theta}, q) < \bar{\theta}$ for $\eta \in \{a, b\}$. It follows

$$\bar{\theta}\left[\frac{p-q}{1-q}\right] > \sum f_\eta g_{\eta 2}\bar{\theta}\left[R_\eta - \frac{q}{1-q}\right] = \bar{\theta}\left[\frac{p-q}{1-q}\right],$$

which again is a contradiction. Hence $\Lambda(q) < \bar{\theta}$ for $q \in Q$. ∎

The implication of Theorem 1 is a level of certain endowment $w \in [0, \theta]$ can be determined such that, given $(\Lambda(q), q)$, the $\xi_\eta(\Lambda(q), q)$ solve (17) and markets clear. Figure (2) plots the function $\Lambda(q)$ for $U(c) = \ln(c)$ and parameter values $f_a = 0.6$, $g_{a1} = 0.3$, and $g_{b1} = 0.6$.[5]

There are three types of separating equilibria: (i) Type a has full consumption insurance, consuming $w = \Lambda(g_{a1})$ and type b agents have partial consumption insurance; (ii) Type b agents have full consumption insurance $w = \Lambda(g_{b1})$ and type a agents have partial consumption insurance; and (iii) Neither type has full consumption insurance, so $q \neq g_{a1}$ and $q \neq g_{b1}$. Notice $q = p$ and $w = \Lambda(p)$ is an example of case (iii) where agents face a budget constraint parallel to and strictly below the average resource constraint.

This formulation of the problem doesn't distinguish between high-risk and low-risk agents. To illustrate this, suppose $q < g_{a1}$ in Figure 1. Then the vertical intercept of the resource constraint is greater than the vertical intercept of the budget constraint or $\frac{\bar{\theta}}{1-p} > \frac{w}{1-q}$ because $\Lambda(q) < \bar{\theta}$; if this did not hold, then either $w > \bar{\theta}$ or else the budget constraint intersects the average resource constraint below the 45-degree line, reversing the definition of which agent type is high risk or low risk. Specifically, given an equilibrium $(q, \Lambda(q))$, if the solution $(\hat{\theta}_1, \hat{\theta}_2)$ to the pair of equations

$$\bar{\theta} \quad = \quad p\hat{\theta}_1 + (1-p)\hat{\theta}_2 \qquad (25)$$

$$\Lambda(q) \quad = \quad q\hat{\theta}_1 + (1-q)\hat{\theta}_2 \qquad (26)$$

[5]The demand functions are $\xi_{\eta 1} = \frac{w g_{\eta 1}}{q}$ and $\xi_{\eta 2} = \frac{g_{\eta 2} w}{1-q}$. Given $q^\star \in Q$, the solution to (28) is

$$w^\star = \Lambda(q^\star) = \bar{\theta}\left[\sum f_\eta \left(\frac{g_{\eta 1}^2}{q^\star} + \frac{g_{\eta 2}^2}{1-q^\star}\right)\right]^{-1}.$$

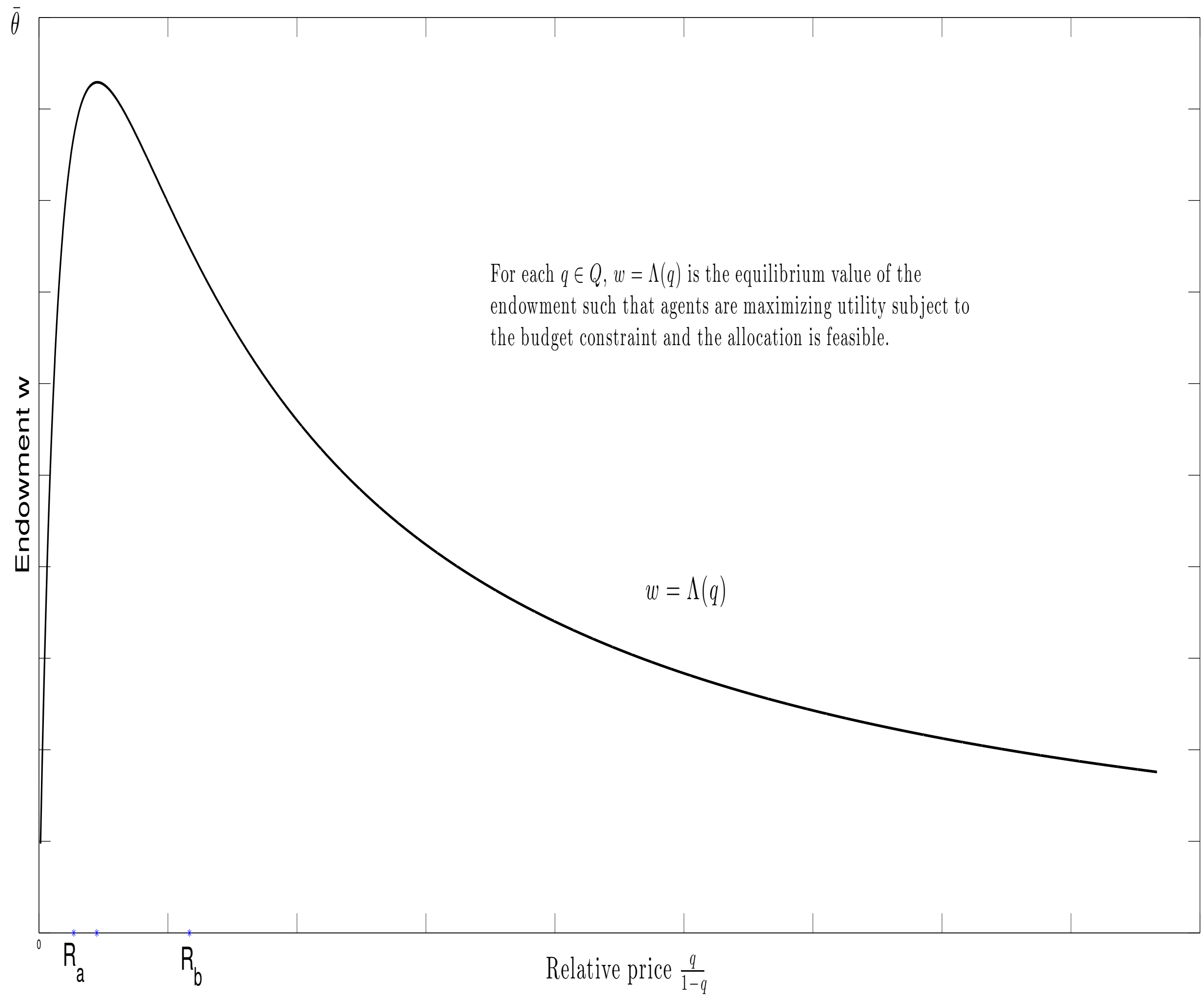

Figure 2: *Equilibrium endowment as a function of the relative price. For a given relative price* $0 < q < 1$, $w = \Lambda(q)$ *is the level of endowment such that agents maximize expected utilility, markets clear and the consumption allocation is feasible.*

has the property $\hat{\theta}_1 > \hat{\theta}_2$, then the definition of high risk and low risk has been reversed. Solving this pair of equations results in

$$\hat{\theta}_2 - \hat{\theta}_1 = \frac{\bar{\theta} - \Lambda(q)}{q - p}. \tag{27}$$

Since $\Lambda(q) < \bar{\theta}$, it follows $q > p$ to ensure type a agents are low risk and type b agents are high risk. If $q = p$, then this pair of equations has no solution because $\Lambda(p) < \bar{\theta}$ and (25) and (26) do not intersect.

For any $q \in Q$, there exists an equilibrium if $w = \Lambda(q)$. For any $w \in [0, \bar{\theta}]$, there may not exist an equilibrium price $q \in Q$ or there may be multiple equilibria. The main result is stated in Theorem 2 but before stating that theorem, the following lemma describes the relationship between $(q, \Lambda(q))$.

Lemma 1 *i) If $0 < q < p$, then type a agents are high risk and type b are the low risk in the private trading equilibrium $(q, \Lambda(q))$.*
(ii) Let q^m denote a solution to $0 = \Lambda'(q)$, where there may be more than one solution. Then $g_{a1} < q^m < g_{b1}$.

Proof.
Part (i) follows from (27) because $\Lambda(q) < \bar{\theta}$ for all $q \in Q$ so, if $\hat{\theta}_2 > \hat{\theta}_1$, where $(\hat{\theta}_1, \hat{\theta}_2)$ solve (25)–(26), then $q \geq p$. By assumption $g_{a1} < g_{b1}$ so, if $\hat{\theta}_2 < \hat{\theta}_1$, the definition of high risk and low risk has been reversed. For part (ii), the function Λ is continuous and differentiable. Differentiating Λ with respect to q

$$\begin{aligned}
\Lambda'(q) &= -\left[\sum f_\eta \sum g_{\eta i} \frac{\partial \xi_{\eta i}(w,q)}{\partial q}\right]\left[\sum f_\eta \sum g_{\eta i} \frac{\partial \xi_{\eta i}(w,q)}{\partial w}\right]^{-1} \\
&= -\left[\sum f_\eta \left(\frac{\partial \xi_{\eta 1}(w,q)}{\partial q} \tau_\eta(q) + g_{\eta 2}\left[\frac{\xi_{\eta 2}(w,q) - \xi_{\eta 1}(w,q)}{1-q}\right]\right)\right]\left[\sum f_\eta \sum g_{\eta i} \frac{\partial \xi_{\eta i}(w,q)}{\partial w}\right]^{-1}.
\end{aligned}$$

The denominator of the right side is strictly positive. Recall $\frac{\partial \xi_{\eta 1}(w,q)}{\partial q} < 0$. If $q \leq g_{a1}$, then $\tau_a(q) \geq 0$ and $\tau_b(q) > 0$. Also $\xi_{a2}(w,q) - \xi_{a1}(w,q) \leq 0$ and $\xi_{b2}(w,q) - \xi_{b1}(w,q) < 0$ so the numerator in brackets is negative, therefore the expression on the right side is positive. Hence $\Lambda'(q) > 0$ for $q \leq g_{a1}$. If $q > g_{b1}$, then $\tau_a(q) < 0$ and $\tau_b(q) \leq 0$ so the term in the numerator in brackets is positive. Also $\xi_{a2}(w,q) - \xi_{a1}(w,q) > 0$ and $\xi_{b2}(w,q) - \xi_{b1}(w,q) \geq 0$ so the right side is negative. Hence $\Lambda'(q) < 0$ for $q \geq g_{b1}$. It follows $g_{a1} < q_m < g_{b1}$.
∎

The following theorem summarizes the main results about the existence of equilibria in the private trading economy.

Theorem 2 *(i) Let* $W = \Lambda(Q)$ *denote the range of* Λ*. For any* $w \in W$*, there exists a private trading equilibrium price* q *where* $q \in Q$*.*

(ii) Let $w^m = \Lambda(q^m)$*, where* q^m *is the solution to* $0 = \Lambda'(q)$*. If there is more than one solution* q^m*, define* w_m *as the maximum value or* $w_m = \max_{q^m} \Lambda(q^m)$*. Then* $W \bigcap (w_m, \bar{\theta}] = \varnothing$*, so if* $w \in (w^m, \bar{\theta}]$ *no private trading equilibrium exists.*

(iii) If $q^m < p$ *for all* q^m*, then, for any* $w \in W$*, there exists a unique equilibrium price* q *such that* $q > p$*. If there is some* q^m *such that* $q^m > p$*, then for any* $w \in [\Lambda(p), w_m]$*, there exists at least two equilibria* (q_1, q_2) *such that* $p < q_1 < q^m$ *and* $q_2 > q_m$*.*

Proof.

For part (i), it follows from the properties of the function Λ established in Theorem 1 the range $W = \Lambda(Q)$ is well-defined and an equilibrium $q \in Q$ exists for any $w \in W$; however it may not be unique. For part (ii), for any $q \in Q$, the equilibrium value of the market endowment satisfies $w = \Lambda(q) \leq w_m$, so the statement follows. For part (iii), the assumption type a agents are low risk requires $q \geq p$. If $q_m < p$ then for any $q \in [p, 1)$, $\Lambda'(q) < 0$. Hence for $w \in (0, w_m]$, there exists a unique equilibrium $q \in [p, 1)$. If there is some q^m such that $p < q_m$, then for each $w \in [\Lambda(p), w_m]$ continuity of Λ implies there exists at least two equilibria q_1, q_2 such that $q_1 < q_m$ with $\Lambda'(q_1) > 0$ and $q_2 > q_m$ with $\Lambda'(q_2) < 0$. ∎

Define the set $\mathcal{P}$ as the set of market-endowment private-trading competitive equilibria, specifically

$$\mathcal{P} \equiv \{(q, w) \in (0,1) \times W \mid q \in (p, 1) \qquad \text{and} \qquad w = \Lambda(q)\}$$

4 Constrained Efficiency of Competitive Equilibria

The objectives of this section are to establish competitive equilibria $\mathcal{P}$ may not be private-trading constrained Pareto efficient, to describe the externality causing the inefficiency, and to describe a simple policy to internalize partly the externality causing the inefficiency. Within the space of separating allocations, the social planning problem (19) is formulated as

$$\mathcal{V}(\Psi) \equiv \max_{\{q \in Q\}} [\Psi V(\Lambda(q), q; a) + (1 - \Psi) V(\Lambda(q), q; b)]. \tag{28}$$

Denote

$$\mu_\eta \equiv \frac{\partial V(w, q; \eta)}{\partial w}.$$

The first-order condition for (28) is

$$0 = \Lambda'(q)\left[\Psi\mu_a + (1-\Psi)\mu_b\right] + \Psi\mu_a[\xi_{a2}(w,q) - \xi_{a1}(w,q)] + (1-\Psi)\mu_b[\xi_{b2}(w,q) - \xi_{b1}(w,q)]$$

where Roy's Identity as been used, which takes the form

$$\frac{\partial V_\eta(w,q)}{\partial q} = -\mu_\eta\left[\xi_{\eta 1}(w,q) - \xi_{\eta 2}(w,q)\right]$$

in this application.

Theorem 3 *The solution $q^\star$ to (19)-(20)) has the following properties.*

(i). If $q^m \leq p$, then $q^\star$ satisfies $q^\star \in (p, g_{b1})$.

(ii) If $q_m < p$, then the solution to (29) is unique.

(iii) If $p < q^m$, then there exists two solutions which cannot be Pareto ordered.

The optimal solution can be implemented by setting a ceiling $\frac{q}{1-q} < R_b$ on the relative price of insurance.

Proof.

If $q^m < p$, then $\Lambda'(q) < 0$ for all $q > p$. Since

$$\xi_{\eta 2}(w,q) - \xi_{\eta 1}(w,q) = \frac{w - \xi_{\eta 1}(w,q)}{1-q},$$

the first-order condition can be expressed as

$$\Lambda'(q) = \left[\frac{\Psi\mu_a}{\Psi\mu_a + (1-\Psi)\mu_b}\right]\left[\frac{\xi_{a1}(w,q) - w}{1-q}\right] + \left[\frac{(1-\Psi)\mu_b}{\Psi\mu_a + (1-\Psi)\mu_b}\right]\left[\frac{\xi_{b1}(w,q) - w}{1-q}\right]. \tag{29}$$

If $q \geq g_{b1}$, then

$$\xi_{\eta 1}(w,q) - w > 0$$

for $\eta = a, b$ and the right side of (29) is positive and the left side is negative, which is a contradiction, hence $q^\star < g_{b1}$. The uniqueness of $q^\star$ when $q^m < p$ follows from the property Λ is decreasing in q for $q > q^m$.

Let $p < q_1 < q^m < q_2$ be two equilibria such that $w = \Lambda(q_1) = \Lambda(q_2)$. By construction, the budget constraint $B(q_1, w)$ intersects $B(q_2, w)$ at w and $B(q_1, w)$ lies above $B(q_2, w)$ for $c_1 > w$ and $c_2 < w$ and lies below for $c_1 > w, c_2 < w$. Since $\xi_{\eta 1}(q, w) > w$ for $q < g_{b1}$, the type b consumption bundle $\xi_b(q_1, w)$ cannot be purchased when $q = q_2$, hence type b agents strictly prefer $B(q_1, w)$. A similar argument can be used to demonstrate type a agents strictly prefer the consumption bundle $\xi_a(q_2, w)$ to $\xi_a(q_1, w)$. Hence the equilibria cannot be Pareto ranked.

∎

4.1 Inefficient Cross-Subsidization

Theorem (3) establishes the optimal solution to the social planning problem (19)-(20) satisfies $q^\star \in (p, g_{b1})$. Competitive equilibria $(q, \Lambda(q))$ such that $q \geq g_{b1}$ are not private-trading constrained efficient. The reason is inefficient cross subsidization. Recall

$$\bar{\theta} - w = \sum f_\eta g_{\eta 2}[\xi_{\eta 1}(w, q) - w][R_\eta - \frac{q}{1-q}]$$

When $q > g_{b1}$, then $R_\eta - \frac{q}{1-q} < 0$ for both types of agents. The expected claim of an agent is $R_\eta(\xi_{\eta 1}(w, q) - w)$ while the payment is $\frac{q}{1-q}(\xi_{\eta 1}(w, q) - w)$. For both types of agents the expected benefits are less than the payments. When $p < q < g_{b1}$, the expected claim for a type $b agent exceeds the cost, while the opposite is true for a type a$ agent. As a result, the private trading constrained efficient outcome has the property type a agents are subsidizing type b agents.

5 Conclusion

Private information creates frictions in trading that are important in understanding the structure of financial markets and risk sharing more broadly. Incentive-efficient allocations in the optimal contracting literature generally require extensive restrictions on the actions of agents, because the individual incentive compatibility constraints create a consumption externality in that the net trades of one agent must be individually incentive compatible with the net trades of other agents. These extensive restrictions on agents' activities may be implausible in decentralized financial markets in which agents have an incentive to eliminate arbitrage profit opportunities and improve risk sharing. An exchange economy with adverse selection and private information is studied under the assumption risk averse agents trade directly in a contingent claims market. Markets are not separated by type, contracts are not exclusive, and agents can enter into private-trading arrangements. The resulting equilibrium allocation is individually incentive compatible, although it is not incentive-efficient. Since agents face the same budget constraint in private-trading markets, but face different endowment distribution risk, agents will execute different net trades depending on type. Essentially agents are trading probability distributions, as in the reinsurance literature. The result that some states are under-insured while others are over-insured for an agent. A price ceiling on the price of insurance is a simple policy to eliminate inefficient cross-subsidization across different types of agents with the result competitive equilibria are private-trading constrained efficient.

Appendix A

To demonstrate not all equilibria in $\mathcal{P}$ are constrained-efficient, I show there may be two equilibria with budget constraints intersecting at the same common endowment (θ_1, θ_2) and these equilibria can be Pareto ranked.

Define the set of of common feasible endowments Θ as

$$\Theta \equiv \{\theta \equiv (\theta_1, \theta_2) \in R_+^2 \mid p\theta_1 + (1-p)\theta_2 \qquad \text{and} \qquad \theta_2 > \theta_1\}.$$

For any $\theta \in \Theta$, define

$$X(\theta) = \{x_\eta = (x_{\eta 1}, x_{\eta 2}) \mid \theta_i + x_{\eta i} \geq 0\}.$$

and define

$$\bar{B}(q,\theta) = \{x \in X(\theta) \mid qx_1 + (1-q)x_2 = 0\}$$

Let

$$x_\eta(q,\theta) = \arg \max_{x \in \bar{B}(q,\theta)} \left\{ \sum_i g_{\eta i} U(\theta_i + x_{\eta i}) \right\}$$

Definition 1 *A common endowment competitive equilibrium is a price $p < q < 1$, endowment $\theta \in \Theta$, and net trades (x_a^e, x_b^e) such that*

$$x_\eta^e = x_\eta(q,\theta)$$

and the allocation is feasible

$$\bar{\theta} = \sum_\eta f_\eta \sum_i g_{\eta i}[\theta_i + x_{\eta i}(q,\theta)].$$

To determine if a common endowment competitive equilibrium exists, let $Q_\eta(\theta)$ denote the (unique) "autarky" price at which an agent of type η has zero demand,

$$Q_\eta(\theta) \equiv R_\eta \frac{U'(\theta_1)}{U'(\theta_2)}, \qquad \eta = a, b.$$

Since $\theta_1 < \theta_2$, it follows $Q_\eta(\theta) > R_\eta, \eta = a, b$ and since $R_b > R_a$ it follows $Q_b(\theta) > Q_a(\theta)$ for all $\theta \in \Theta$. Define

$$q_\eta^a(\theta) \equiv \frac{Q_\eta(\theta)}{1 + Q_\eta(\theta)}$$

The demand functions x_η, $\eta = a, b$ have the following properties.

(i) An agent of type η insures (does not insure, takes on risk) if and only if $\frac{q}{1-q} < Q_\eta(\theta)$ ($\frac{q}{1-q} = Q_\eta(\theta)$, $\frac{q}{1-q} > Q_\eta(\theta)$). That is

$$x_{\eta 1}(q, \theta) \gtreqqless 0 \qquad \text{as} \qquad \frac{q}{1-q} \lesseqqgtr Q_\eta(\theta).$$

(ii) An agent of type η insures partially (fully, more than fully) if and only if $\frac{q}{1-q} > R_\eta$ ($\frac{q}{1-q} = R_\eta$, $\frac{q}{1-q} < R_\eta$). That is

$$\theta_1 + x_{\eta 1}(q, \theta) \lesseqqgtr \theta_2 + x_{\eta 2}(q, \theta) \qquad \text{as} \qquad \frac{q}{1-q} \gtreqqless R_\eta.$$

An immediate consequence of (i) and (ii) is

(iii) If $Q_a(\theta) < R_b$, then full consumption insurance for high risk (type b) agents implies that low risk type a agents take on risk. That is,

$$Q_a(\theta) < R_b \Longrightarrow x_{a1}(g_{b1}, \theta) < 0.$$

There are two cases,

$$\text{Case I:} \qquad R_a \quad < \quad R_b < Q_a(\theta) < Q_b(\theta), \tag{30}$$

$$\text{Case II:} \qquad R_a \quad < \quad Q_a(\theta) \leq R_b < Q_b(\theta). \tag{31}$$

The results for competitive equilibria starting from a common endowment $\theta \in \Theta$ are studied in Theorem 1 of Labadie [2009], which is summarized below.

Theorem 4 *Under assumptions (1)–(2),*
(i) if $Q_a(\theta) > R_b$ (Case I), there exists at least one equilibrium with $\frac{q}{1-q} \in (\bar{R}, R_b)$ and at least one with $\frac{q}{1-q} \in (Q_a(\theta), Q_b(\theta))$, and there are no equilibria with $\frac{q}{1-q} \in [R_b, Q_a(\theta)]$;
(ii) if $R_b = Q_a(\theta)$, then $\frac{q}{1-q} = R_b$ is an equilibrium;
(iii) if $Q_a(\theta) < R_b$ (Case II), then in any equilibrium $\frac{q}{1-q} \in (\bar{R}, Q_a(\theta)) \cup (R_b, Q_b(\theta))$.

The proof is omitted. See Labadie [2009].

I now establish the connection between the common endowment equilibria and the set of competitive equilibria $\mathcal{P}$. Associated with each $(q, w) \in \mathcal{P}$ is an endowment point (θ_1, θ_2) on the resource constraint, subject to some restrictions. Let $\theta \equiv (\theta_1, \theta_2) \in \Theta$ be the solution

$$\begin{aligned} \bar{\theta} &= p\theta_1 + (1-p)\theta_2 & (32) \\ \Lambda(q) &= q\theta_1 + (1-q)\theta_2 & (33) \end{aligned}$$

To ensure $\theta_1, \theta_2) \in \Theta$ requires $\frac{w}{1-q} > \frac{\bar{\theta}}{1-p}$ and $\frac{w}{q} < \frac{\bar{\theta}}{p}$.[6]. The budget sets $\bar{B}(\theta, q)$ and $B(w, q)$ then intersect at the point (θ_1, θ_2). The next step is establish the values of $\theta \in \Theta$ satisfying Case I and Case II in Theorem (3). This requires determining the allocation $\theta \in \Theta$ satisfying the condition in part (ii) of Theorem (3). Let $\theta^a \equiv (\theta_1^a, \theta_2^a)$ solve the pair of equations

$$\begin{aligned} \bar{\theta} &= p\theta_1 + (1-p)\theta_2 & (34) \\ R_b &= Q_a(\theta) = R_a \frac{U'(\theta_1)}{U'(\theta_2)}. & (35) \end{aligned}$$

Then $q = g_{b1}$ is an equilibrium when the endowment is θ^a. Define the set Θ_I as

$$\Theta_1 = \{(\theta_1, \theta_2) \in \Theta \mid \theta_1 \leq \theta_1^a \quad \text{and} \quad \theta_2 \geq \theta_2^a\}$$

and define the set Θ_{II}

$$\Theta_2 = \{(\theta_1, \theta_2) \in \Theta \mid \theta_1 > \theta_1^a \quad \text{and} \quad \theta_2 < \theta_2^a\}$$

Case I in (30) corresponds to $(\theta_1, \theta_2) \in \Theta_1$ while Case II in (31) corresponds to $(\theta_1, \theta_2) \in \Theta_2$. Set $q = g_{b1}$ and define $w = \Lambda(g_{b1})$. The common endowment point corresponding to this equilibrium is the solution to the pair of equations

$$\begin{aligned} \bar{\theta} &= p\theta_1 + (1-p)\theta_2 & (36) \\ \Lambda(g_{b1}) &= g_{b1}\theta_1 + g_{b2}\theta_2 & (37) \end{aligned}$$

Denote the solution as $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$.

Proposition 1 *The solution θ^a to (34)-(35) solves the system (36)-(37).*

PROOF.
Define $w^a \equiv g_{b1}\theta_1^a + g_{b2}\theta_2^a$. Then $\xi_{a1}(w_a, g_{b1}) = \theta_1^a$, $\xi_{a2}(w_a, g_{b1}) = \theta_2^a$ and $\xi_{b1}(w^a, g_{b1}) = \xi_{b2}(w^a, g_{b1})$. This allocation satisfies the market-clearing condition (20) setting $\Lambda(g_{b1}) = w^a$. Hence $\theta^a = \hat{\theta}$. ∎

[6]Prove this

For each $\theta \in \Theta_1$, there are at least two equilibria such that $q_1 \in (p, g_{b1})$ and $q_2 \in \left(\frac{Q_a(\theta)}{1+Q_a(\theta)}, \frac{Q_b(\theta)}{1+Q_b(\theta)}\right)$.$q_1 \in (p, g_{b1})$ and $q_2 \in \left(\frac{Q_a(\theta)}{1+Q_a(\theta)}, \frac{Q_b(\theta)}{1+Q_b(\theta)}\right)$. Let

$$\begin{aligned} w_1 &= q_1\theta_1 + (1-q_1)\theta_2 \\ w_2 &= q_2\theta_1 + (1-q_2)\theta_2 \end{aligned}$$

It follows $w_1 > w_2$ because $q_1 < q_2$ and $\theta_2 > \theta_1$. The two budget constraints intersect at the point θ_1, θ_2. For q_1, the type b, agent has net demands such that $x_{b1}(q_1, \theta) > 0$ and $x_{a2}(q_1, \theta) < 0$ such that $\theta_1 + x_{b1}(q_1, \theta) > \theta_2 + x_{b2}(q_1, \theta)$, so his consumption is below the 45-degree line on the budget constraint.The type b agent has net demands such that $x_{b1}(q_2, \theta) > 0$ and $x_{a2}(q_2, \theta) < 0$ such that $\theta_1 + x_{b1}(q_2, \theta) < \theta_2 + x_{b2}(q_2, \theta)$, so his consumption is on the budget constraint on the segment (θ_1, θ_2) and (w_2, w_2). For the type b agent, his consumption $(\theta_1 + x_{b1}(q_2, \theta), \theta_2 + x_{b2}(q_2, \theta)$ is contained in his budget set $\hat{B}(q_1, \theta)$ so he is better off with consumption$(\theta_1 + x_{b1}(q_1, \theta), \theta_2 + x_{b2}(q_1, \theta))$.

For q_1, the net demands for a type a agent satisfy $x_{a1}(q_1, \theta) > 0$ and $x_{a2}(q_1, \theta) < 0$ such that $\theta_1 + x_{a1}(q_1, \theta) < \theta_2 + x_{a2}(q_1, \theta)$. Hence the type a agent's consumption lies along the budget constraint on the line segment between the point (θ_1, θ_2) and (w_1, w_1). For the equilibrium with price q_2, observe the net demands for a type a agent satisfy $x_{a1}(q_2, \theta) < 0$ and $x_{a2}(q_2, \theta) > 0$ such that $\theta_1 + x_{a1}(q_2, \theta) < \theta_2 + x_{a2}(q_2, \theta)$. Hence the type a agent's consumption lies along the budget constraint on the line segment above the point (θ_1, θ_2). For the type a agent, define the expected utility of his consumption in equilibrium (q_1, w_1) as

$$\bar{U}_a = g_{a1}U(\theta_1 + x_{a1}(q_1, \theta) + g_{2a}U(\theta_2 + x_{a2}(q_1, \theta))$$

Define the compensated demand curve $h_a(q_2, \bar{U}_a) \equiv (h_{a1}(q_2, \bar{U}_a), h_{a2}(q_2, \bar{U}_a)$. Observe $q_2 h_{a1}(q_2, \bar{U}_a) + (1 - q_2)h_{a2}(q_2, \bar{U}_a) > w_2$, hence the type a is better off in equilibrium q_1, w_1. Hence the competitive equilibrium q_1, w_1 Pareto dominates q_2, w_2.

A similar argument can be constructed for a common endowment $\theta \in \Theta_2$ for which there are at least two equilibria such that $q_1 \in (p, \frac{Q_a(\theta)}{1+Q_a(\theta)})$ and $q_2 \in \left(R_b, \frac{Q_b(\theta)}{1+Q_b(\theta)}\right)$.Hence attention will be focused on $q \in (p, g_{b1})$ for all $\theta \in \Theta$.

Appendix B

The equivalence between the original problem (11)-(14) and the modified problem (19)-(20) is summarized in Theorem 3; the proof follows the proof of Lemma 1 in FGT.

Theorem 5 *Let* $(w^\star, q^\star)$ *be a solution to (19)-(20) and let* $\{\xi_a(w^\star, q^\star), \xi_b(w^\star, q^\star)\}$ *be a solution to (17), given* $(w^\star, q^\star)$*, where* $(w^\star, q^\star)$ *is a private trading competitive equilibrium. Then* $\{c(a), c(b)\}$ *defined by*

$$c(\eta) = (\xi_{\eta 1}(w^\star, q^\star), \xi_{\eta 2}(w^\star, q^\star)) \qquad \textit{for all} \qquad \eta \in \{a, b\} \tag{38}$$

is a solution to (11)–(14). Conversely, let $\{c(a), c(b)\}$ *be a solution to (11)–(14) with the property it is a separating allocation. Then there exists a* $q \in Q$ *and* $w = \Lambda(q) \in \bar{\Theta}$ *solving (19) subject to (20) if* $\xi_\eta(w^\star, q^\star) = c(\eta)$ *such that* $\{\xi_a(w^\star, q^\star), \xi_b(w^\star, q^\star)\}$ *solve (17).*

The proof of Theorem 1 is a straightforward application of the proof to Lemma 1 in FGT.

Proof.

The first step is to show the solution to (11)–(14), denoted $\{c^\star(a), c^\star(b)\}$ can be implemented for some w and q satisfying (20), in that $\{c^\star(a), c^\star(b)\}$ would solve (17). Start with a solution $\{c(a), c(b)\}$ to (11) that is a separating allocation and let $q \in Q$ be the associated equilibrium price, as in Definition 2. The incentive compatibility constraints (13)-(14) can be written

$$\sum_i g_{\eta i} U(c_i(\eta)) \geq \hat{V}(qc_1(h) + (1-q)c_2(h), q; \eta)$$

for all $h \in \{a, b\}$ and each η. From the definition of $\hat{V}$ in (8) it follows

$$\hat{V}(qc_1(\hat{h}(\{c(a), c(b)\}, q; \eta)) + (1-q)c_2(\hat{h}(\{c(a), c(b)\}, q; \eta)), q; \eta) \geq \hat{V}(qc_1(h) + (1-q)c_2(h), q; \eta) \qquad \text{for all } h \in \{a, b\}$$

which is equivalent to

$$\hat{V}(\{c(a), c(b)\}, q; \eta) \geq \hat{V}(\max[qc_1(a) + (1-q)c_2(a), qc_1(b) + (1-q)c_2(b)], q; \eta) \tag{39}$$

Since $\hat{V}$ is strictly increasing in its first argument, it follows

$$qc_1(\eta) + (1-q)c_2(\eta) = \max_{h \in \{a,b\}} [qc_1(h) + (1-q)c_2(h)]$$

or

$$qc_1(a) + (1-q)c_2(a) = qc_1(b) + (1-q)c_2(b). \tag{40}$$

Denote this value as $\hat{w}$. The next step is to use $\hat{w}$ and q in (17) and show $\{c(a), c(b)\}$ solve (17). Since the equality in (40) is a property of the solution to (8), then this implies (19) if $\xi(w, q; \eta) = c(\eta)$. The allocation $\{c(a), c(b)\}$ satisfies the social planner's feasibility constraint (12) and is still a solution under the restriction

$$\hat{h}(\{c(a), c(b)\}, q; a) = \hat{h}(\{c(a), c(b)\}, q; b)$$

which is implicit in (17). Since $\{c(a), c(b)\}$ is a solution to (11), it follows $\{c(a), c(b)\}$ can be implemented by the appropriate choice of w, q. Moreover

$$\hat{V}(\{c(a), c(b)\}, q; \eta) = V(w, q; \eta)$$

where $\hat{w}$ is defined in (40).

The second step is to take $(w^\star, q^\star)$ solving (19)–(20) and show the $\{c(a), c(b)\}$ given by

$$c(\eta) = (\xi_{\eta 1}(w^\star, q^\star), \xi_{\eta 2}(w^\star, q^\star)),$$

where $\xi_\eta(w^\star, q^\star)$ solve (17), are feasible and solve (11). Given (w, q), the allocation $\{c(a), c(b)\}$ will solve (17) for any η and, because $\{c(a), c(b)\}$ satisfies (12) this implies (20) if $\xi_{\eta i}(w, q) = c_i(\eta)$ for all η and i. By assumption $\{c(a), c(b)\}$ is a solution to (8) and is still a solution under the additional constraint $\hat{h}(\{c(a), c(b)\}, q; a) = \hat{h}(\{c(a), c(b)\}, q; b)$. Hence any solution $\{c(a), c(b)\}$ to (11) that is a separating allocation may be implemented through the appropriate choice of w and q. Moreover the values of the maximand in (11) and (19) are equal for these parameter values; hence the maximum in (19) is at least as large as the one in (11).

Suppose $(w^\star, q^\star)$ solve problem (19) and let $\{c(a), c(b)\}$ be given by $c_i(\eta) = \xi_{\eta i}(w^\star, q^\star)$, where $\{\xi_a(w^\star, q^\star), \xi_b(w^\star, q^\star)\}$ is the solution to (17) given $(w^\star, q^\star)$. The next step is to determine if $\{c(a), c(b)\}$ is feasible for (11), so it satisfies (12). If (20) is satisfied by $\{\xi_a(w^\star, q^\star), (\xi_b(w^\star, q^\star)\}$ and $c(\eta) = \xi_\eta(w^\star, q^\star)$ then clearly (12) follows from (20). Since the budget constraint in problem (19) is binding, it follows

$$q^\star c_1(a) + c_2(a) = q^\star c_1(b) + c_2(b) = w^\star$$

so

$$\hat{V}(\{c(a), c(b)\}, q^\star; \eta) = V(w^\star, q^\star; \eta) \qquad \text{for all} \qquad \eta \in \{a, b\},$$

because with certain endowment $w^\star$, given this relative price $q^\star$ and private markets, a consumer does as well reporting truthfully

$$\eta = \hat{h}(\{c(a), c(b)\}, q; \eta)$$

as he does by lying,

$$h = \hat{h}(\{c(a), c(b)\}, q; \eta) \qquad h \neq \eta.$$

The incentive compatibility constraints (13)–(14) follow from the property $\{\xi_a(w^\star, q^\star), \xi_b(w^\star, q^\star)\}$ solve (17). Hence, if $(w^\star, q^\star)$ solve (19), then the induced allocations are feasible for problem (11). Also, the maximands are equal, and the maximum in (11) is at least as large as the one in (19).

Hence, if the allocation is separating, the maximums in (11) and (19) coincide. Any separating allocation $\{c(a), c(b)\}$ solving (11) induces a $(w^\star, q^\star)$ solving (19). ∎

Appendix C: Discussion of Market Structure

The market structure assumed in Section 3 is discussed. Agents trade directly among themselves after receiving the announcement-based endowment and it is assumed there is no financial intermediary. There are two possibilities for the organization of competitive markets when agents trade directly among themselves: (i) There is either a single market in which all claims contingent on realization $i \in \{1, 2\}$ trade for the identical price vector $q_h, 1 - q_h$, or else (ii) there are separate markets A and B, where contingent claims trade at prices $\{(q_a, 1 - q_q), (q_b, 1 - q_b)\}$ and $q_a \neq q_b$. The social planner offers a consumption vector $c \in I$. An agent who announces he is type η receives $c(\eta)$. If the agent subsequently trades in market $h, h \in \{A, B\}$, he faces a budget constraint

$$q_h c_{\eta 1} + (1 - q_h) c_{\eta 2} = q_h x_{\eta 1} + (1 - q_h) x_{\eta 2} \qquad \eta, h \in \{a, b\}.$$

Non-exclusivity of contracts implies agents can enter into multiple trades and current trades are not conditional on previous trades. The first issue is whether markets can be separate when there is private information about type and non-exclusivity in contracts. An agent can enter into contingent claims markets multiple times as long as the budget constraint is satisfied.

There are three possibilities for $c \in I$: (i) both types have full insurance; (ii) one type has full insurance while the other does not; (iii) neither type has full insurance. To start, suppose there are separate markets.

If an agent of type η self selects into market η, then by symmetry all agents of type η self select into market η. The implications of separate markets are derived for the three cases.

Case i: Both types have full insurance at consumption levels $\bar{c}_a, \bar{c}_b$. If $\bar{c}_a \neq \bar{c}_b$, then both agents will announce the type providing the higher level of certain consumption, so incentive compatibility requires $\bar{c}_a = \bar{c}_b$, which is feasible if $\bar{c}_\eta = \bar{\theta}$ (if less than $\bar{\theta}$ there are unused resources). If the social planner offers each agent $\bar{\theta}$ for both types and states, then agents are identical across all states and types so there is no idiosyncratic risk, no trading and hence no side markets are formed.

Case ii: One type has full insurance at $\bar{c}_\eta$ while the other does not, so $c_{h1} \neq c_{h2}, h \neq \eta$. Assume $\bar{c}_\eta \neq \bar{\theta}$. If only type h agents self select into market h, then the equilibrium price satisfies $q_h = g_{h1}$ and type h agents will trade such that they achieve full consumption insurance equal to

$$\hat{c}_h = g_{h1}c_{h1} + g_{h2}c_{h2}.$$

If $\hat{c}_h \neq \hat{c}_\eta$, then the final consumption allocation is not incentive compatible. If, as assumed, a type h agent self-selects into market h, then $\hat{c}_h \geq \hat{c}_\eta$, otherwise he would have announced type η. It follows, if type h agents obtain $\hat{c}_h > \hat{c}_\eta$, then type η agents would also announce type h in anticipation of the higher consumption after trade and the equilibrium price cannot satisfy $q_h = g_{h1}$. If $\bar{c}_\eta = \bar{\theta}$, then after trading with other agents of type $h \neq \eta$, a type h agent has final consumption $\hat{c}_h = \bar{\theta}$. Type η agents have no incentive to announce they are type h and type h agents are indifferent between announcing type η and type h.

Case iii: Neither agent has full insurance. There are two sub-cases. If $c_i \equiv c_{\eta i} = c_{hi}$, as would be the case if agents receive identical contingent endowments (θ_1, θ_2), and if markets were separate, then all agents will engage in arbitrage trading because, for any (c_1, c_2), $0 \neq c_1[q_a - q_b]$ and an arbitrage opportunity exists. Trading across markets will eliminate the arbitrage opportunity until $q_\eta = q_h$. In the other case, if agents receive different endowments, $c_{ai} \neq c_{bi}$, then agents will engage in trade to eliminate arbitrage profits. Moreover, if the value of the consumption allocations $c \in I$ in the side markets is different,

$$qc_{a1} + (1-q)c_{a2} \neq qc_{b1} + (1-q)c_{b2},$$

then an agent will announce the type with the highest value of consumption in side markets.

To summarize, unless one or both types have full consumption insurance, where certain consumption is equal to $\bar{\theta}$, there will be a single market for contingent claims. Moreover, if the allocation $c \in I$ has the property neither type has full insurance, then the consumption allocation for each type must have the same value in side markets.

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