Accounting for the Sources of Macroeconomic Tail Risks

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PRELIMINARY AND INCOMPLETE

Abstract

We empirically examine the sources of aggregate tail risks. Using data on sectoral employment from 1947 to the present, in conjunction with a multi-industry model of the type introduced in Long and Plosser (1983), we calculate the contribution of industries' disturbances to aggregate output and employment tail risks. Our main finding is that durable goods manufacturing accounts for a plurality of measured tail risk. A pure statistical model of industry-level employment growth understates the importance of the construction and finance, insurance, and real estate industries.

1 Introduction

Aggregate activity exhibits tail risks. That is, the distribution of aggregate fluctuations displays both negative skew and has fatter tails than that of a normally distributed random variable. Understanding these tail risks are important: As forcefully argued by Barro (2009), higher-order moments of aggregate activity are critical in assessing the utility cost of macroeconomic fluctuations. Moreover, allowing for non-normal innovations are of first-order importance in forecasting; see for example, Cúrdia, Del Negro, and Greenwald (2014). In this paper, we present an accounting framework with which to decompose the sources of macroeconomic tail risks, and apply this framework to data on industry-level employment growth rates. Our contribution is to empirically investigate whether these higher moments have origins in particular sectors.

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We begin the paper, in Section 2, by exploring the distribution of industries' employment growth rates. As we document, there are substantial differences in the extent to which industries' employment growth rates depart from the normal distribution. A few sectors— e.g. Education and Health and Local Government—have growth rates that cannot be statistically distinguished from the normal distribution. For other industries— most notably Nondurable Manufacturing, Transportation, and Business Services—employment growth rates are highly kurtotic and negatively skewed. According to a simple accounting exercise, based on expansions of the third and fourth moments of the sum of industries' employment growth rates, durable manufacturing and business services employment together account for almost half of aggregate employment tail risk.

Due to input-output linkages, productivity and preference shocks in an individual industry impact employment in other industries. In Section 3, we apply the approach introduced in Foerster, Sarte, and Watson (2011) to recover the underlying productivity shocks from data on industries' employment growth rates, accounting for the correlation in activity, across industries, that is induced by input-output linkages. We then use the distribution of filtered productivity shocks to perform two accounting exercises, each highlighting different aspects of industries' contributions to aggregate fluctuations' departures from normality. In the first exercise, we compute the contribution of the independent component of industry-specific productivity shocks to aggregate skewness and kurtosis. For a given skewness or kurtosis, industries that have high-variance, or industries that comprise a large fraction of aggregate output, have a larger contribution to aggregate tail risk. A second measure captures not only how volatile industries' productivity shocks are, but also how correlated the industry's productivity shocks are to the productivity shocks in other industries. In line with the statistical analysis in Section 2, not only Durables Manufacturing employment but also Durables Manufacturing productivity growth is identified as a key contributor to aggregate employment tail risk. For other sectors, however, the results change significantly: Our benchmark calibration also identifies the productivity shocks in the construction industry as important, and also, to a lesser degree, those in the FIRE industry. In contrast, Business Services productivity shocks are not found to have a significant effect. We show that even after accounting for input-output linkages, industry-specific shocks exhibit significant comovements that are important for understanding the aggregate behavior.

Based on our calibration results we ask two questions: (1) What drives the differences between the inferred productivity shocks and employment growth? (2) To what extent does a common factor account for aggregate tail risks? We find for (1) that when consumers' demand and labor supply elasticities are low, inferred productivity growth resembles employment growth more closely. The relative importance of sectors is, however, not invariant to these preference parameters as the precise effects depend on sectoral factor shares. Regarding (2), the estimated relevance of a common factor is sensitive to our parameterization of consumers' preferences. When the intratemporal elasticity of substitution and of labor supply are relatively low, a single factor explains both aggregate employment growth well over time and also captures the kurtosis of employment growth well. Skewness is not as easily captured by an aggregate factor: several industries explain the substantial residual skewness contributions in the structural decomposition.

This paper contributes primarily to the literature, initiated by Long and Plosser (1983), which hypothesizes that localized, firm-specific or industry-specific disturbances shape aggregate fluctuations. Our paper is closest to Carvalho and Gabaix (2013) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). The latter paper contains a theoretical exploration of the relationship between the distributions of industry-specific productivity shocks and aggregate output. The authors provide necessary and sufficient conditions for idiosyncratic, industry-specific productivity shocks to engender macroeconomic tail risks.[1] Our paper also closely relates to Carvalho and Gabaix (2013). This paper empirically demonstrates that the long-run evolution of aggregate volatility, both the great diversification and its reversal, is attributable to long-run changes in sectoral composition of the economy. Unlike Carvalho and Gabaix (2013), which focuses on the forces that shape the standard deviation of GDP growth, we consider aggregate volatility that more specifically relates to large economic downturns.

In ongoing work, we exploit changes in the input-output structure over the post-war period to assess whether the documented tail behavior can be attributed to economic fundamentals via changing input-output linkages. Similar to Carvalho and Gabaix (2013), we aim to explain the aggregate tail-behavior with economic fundamentals, namely changes in the industry structure and industry linkages.

2 Statistical Decompositions

Our paper applies data from the BLS Current Employment Statistics program. This dataset contains data on industries' employment, for all non-farm industries, for each quarter between 1947 and 2014. We add data on agricultural employment by interpolating annual agricultural employment growth from the Bureau of Economic Analysis. We then remove an Hodrick-Prescott trend computed with the standard quarterly smoothing parameter of

[1] A necessary condition for kurtotic aggregate fluctuations, in Acemoglu, Ozdaglar, and Tahbaz-Salehi, is a skewed size distribution. Earlier papers introduce models in which, as a result of extreme complementarities in production, idiosyncratic shocks to equally sized firms yield fat-tailed aggregate output distributions; see for example Bak et al. (1993).

1,600 from the log industry employment and aggregate the data up with time-varying weights based on the original employment series.[2]

Table 1 presents the standard deviation, skewness, and excess kurtosis of quarter-over-quarter employment growth, both for each individual industry in our sample, and for the economy as a whole. There are substantial cross-industry differences in the distribution of employment growth. Agriculture, Mining/Petroleum Extraction, Construction and Durable Manufacturing have high-variance employment growth. Growth for most industries displays excess kurtosis and negative skewness, with Mining/Petroleum Extraction and Information with the highest excess kurtosis.

Industry	sd_i	$\mathbb{S}_i$	$\mathbb{K}_i$
Agriculture	0.021	-1.77*	9.09*
Mining/Petroleum Extraction	0.026	1.30	18.49*
Construction	0.015	-0.18	1.38*
Durable Manufacturing	0.017	-0.62*	2.69*
Nondurable Manufacturing	0.007	-1.50*	7.58*
Wholesale	0.005	-0.42*	1.03*
Retail	0.005	-0.34*	0.28
Transport.	0.008	-0.54*	0.95*
Utilities	0.007	-0.47	3.05*
Information	0.012	0.69	26.60*
FIRE	0.003	-0.76*	1.27*
Business Services	0.006	-1.21*	2.76*
Education/Health	0.003	-0.51*	2.10*
Leisure and Accommodation	0.005	0.04	1.62
Other Services	0.004	-0.56*	1.14*
Federal Government	0.011	-0.68	9.96*
State Government	0.005	-0.16	2.25*
Local Government	0.004	-0.04	4.03*
Total	0.006	-0.53*	2.70*

Table 1: Employment growth rate standard deviation, skewness, and excess kurtosis.
Notes: Stars indicate statistical significance at the 5 percent level, resulting from boot-strapped confidence intervals under *iid* sampling.

In each quarter, aggregate employment growth is the sum of employment growth in each of the economy's constituent industries. As a result the skewness and excess of kurtosis of aggregate employment growth equals the following combination of the moments

[2] Figure 3 in the Appendix shows in quantile plots that this procedure does increase measured kurtosis slightly , but the quality of our approximation is equally good for the raw or for the data detrended at the industry level, as shown in the lower panels for our baseline calibration.

of the distribution of industries' employment growth

$$\mathbb{S} = \sum_i \mathbb{S}_i^{tot} = \sum_i \mathbb{S}_i^d + \sum_i \mathbb{S}_i^{corr} \tag{1}$$

$$\mathbb{K} = \sum_i \mathbb{K}_i^{tot} = \sum_i \mathbb{K}_i^d + \sum_i \mathbb{K}_i^{corr} \tag{2}$$

In Equations 1 and 2, $\mathbb{S}_i^{tot}$ measures the contribution of industry i to aggregate employment growth skewness.

If industry employment growth rates are independent of one another, the contribution of industry i to aggregate skewness and kurtosis equals

$$\mathbb{S}_i^d \equiv \frac{\omega_i^3 \left(\mathbb{V}_i\right)^{3/2}}{\mathbb{V}^{3/2}} \mathbb{S}_i \text{ and } \mathbb{K}_i^d \equiv \frac{\omega_i^4 \mathbb{V}_i^2}{\mathbb{V}^2} \mathbb{K}_i, \tag{3}$$

respectively, where $\mathbb{V}_i$, $\mathbb{S}_i$, and $\mathbb{K}_i$ are the variance, skewness, and kurtosis of the employment growth rates of industry i, ω_i is the employment share of industry i, and $\mathbb{V}$ is the variance of aggregate employment growth. For a given skewness or kurtosis, high-variance, high-employment-share industries contribute relatively more to high aggregate employment's higher-order moments. The sum of $\mathbb{S}_i^d$ across industries would equal aggregate employment skewness if employment growth rates were independent. According to these measures employment growth in durable goods manufacturing accounts for roughly one fourth of aggregate employment's skewness and one half of its excess kurtosis.

If industry employment growth rates are correlated, moments of the distribution of aggregate employment will also depend on the correlation of industries' employment growth rates. To capture the contribution of these correlated growth rates, we define a second set of measures:[3]

$$\mathbb{S}_i^{tot} = \sum_{j,k,l} \frac{1}{3} \cdot \left(\mathbf{1}_{i=j} + \mathbf{1}_{i=k} + \mathbf{1}_{i=l}\right) \cdot \omega_j \omega_k \omega_l \cdot \frac{\mathbb{E}\left[\Delta \log N_j \Delta \log N_k \Delta \log N_l\right]}{\mathbb{V}^{3/2}}, \tag{4}$$

and

$$\begin{aligned} \mathbb{K}_i^{tot} = &\sum_{j,k,l,m} \frac{1}{4} \left(\mathbf{1}_{i=j} + \mathbf{1}_{i=k} + \mathbf{1}_{i=l} + \mathbf{1}_{i=m}\right) \cdot \omega_j \omega_k \omega_l \omega_m \cdot \frac{\mathbb{E}\left[\Delta \log N_j \Delta \log N_k \Delta \log N_l \Delta \log N_m\right]}{\mathbb{V}^2} \\ &- 3\frac{\kappa(i,\Sigma)}{\mathbb{V}^2}, \end{aligned} \tag{5}$$

[3]We assume that industry employment growth has been demeaned in what follows.

where $\mathbb{V} = \mathbb{E}[(\sum_i \omega_i \Delta \log N_i)^2]$ is the variance of aggregate employment growth and the constant term $\kappa(i, \Sigma)$ sums up to $\mathbb{V}$.[4] The idea behind these measures may be best explained by an analogy with a variance decomposition: In the presence of non-zero covariance terms, a variance decomposition is exact only if it includes covariance terms. In our methodology, we would attribute the covariance between industries I and J with weights $\frac{1}{2}$ and $\frac{1}{2}$ to the variance contribution from industry I and J. For higher-order moments, we assign weights to industries according to how often an industry's contribution shows up in the skewness or kurtosis component. Put differently, when the third moment is based on the second power of industry I's employment growth and the first power of industry J's, we assign weights $\frac{2}{3}$ and $\frac{1}{3}$ and we would assign equal weights to industries I, J, K if the term in question were $\mathbb{E}[\Delta \log N_I \Delta \log N_J \Delta \log N_K]$.

The two measures are presented in the second and fourth columns of Table 2. Overall, the three industries that contribute most to aggregate tail risk are durable goods, non-durable goods, and construction. A number of services industries including business services also contribute sizable amounts to aggregate kurtosis and business services matters also for aggregate skewness. The most pronounced difference, among the two measures, is observed for the durable goods industry, pointing to an important correlation between the industries employment growth rates. In the next section, we relate this correlation both to correlated shocks and input-output linkages.

3 Structural Decompositions

Employment in industry i may fluctuate due to events that can be traced back to other industries in the economy. Productivity shocks in industry j will—to the extent that industries i and j are related through input output linkages—affect employment in industry i. To take these induced correlations into account, we will recover industries' productivity shocks from data on industry employment, and then compute the contribution of industries' productivity shocks to aggregate employment volatility.

[4] The constant term is is given by

$$\kappa(i,\Sigma) \equiv \Sigma_{i,i}^2 + \sum_j \left(\mathbf{1}_{i\neq j}\Sigma_{i,i}\Sigma_{j,j} + \sum_k \left(\mathbf{1}_{i\neq k}\Sigma_{j,j}\Sigma_{i,k} + \mathbf{1}_{j\neq k}\Sigma_{i,i}\Sigma_{j,k} + \sum_\ell \mathbf{1}_{i\neq j}\mathbf{1}_{k\neq \ell}\Sigma_{i,j}\Sigma_{k,\ell} \right) \right)$$
$$= \sum_j \left(\sum_k \left(\sum_\ell \mathbb{V}_{i,j}\mathbb{V}_{k,\ell} \right) \right). \tag{6}$$

Industry	$\mathbb{S}^d$	$\mathbb{S}^{tot}$	$\mathbb{K}^d$	$\mathbb{K}^{tot}$
Agriculture	-0.01	0.01	0.01	0.05
Mining/Petroleum Extraction	0.00	0.00	0.01	0.11
Construction	-0.00	-0.05	0.01	0.21
Durable Manufacturing	-0.28	-0.21	1.92	1.23
Nondurable Manufacturing	-0.02	-0.06	0.02	0.32
Wholesale	-0.00	-0.02	0.00	0.07
Retail	-0.00	-0.04	0.00	0.12
Transport.	-0.00	-0.03	0.00	0.14
Utilities	-0.00	-0.00	0.00	0.03
Information	0.00	-0.02	0.00	0.12
FIRE	-0.00	-0.02	0.00	0.03
Business Services	-0.01	-0.06	0.01	0.12
Education/Health	-0.00	-0.01	0.00	0.04
Leisure and Accommodation	-0.00	-0.03	0.00	0.06
Other Services	-0.00	-0.01	0.00	0.02
Federal Government	-0.00	-0.01	0.01	0.04
State Government	-0.00	0.01	0.00	-0.01
Local Government	0.00	0.02	0.00	0.01
Total	-0.33	-0.53	1.98	2.70
Actual (employment)		-0.53		2.70

Table 2: Statistical decompositions of employment growth skewness and kurtosis.

Notes: Due to approximation error, differences between the industry total and actual employment growth may exist. Rescaled employment is scaled so that average industry-wide employment shares correspond to the model-implied shares.

3.1 Model

3.1.1 Environment

The model broadly follows that in Foerster, Sarte, and Watson (2011), with minor modifications to allow for consumer durables. The aim of this model is to recover industries' productivity shocks from data on industry employment.

The economy consists of N perfectly competitive industries and a representative consumer. The consumer supplies labor and consumes output produced by each of the industries.

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log \left[\left[\sum_{J=1}^{N} \omega_J^{\frac{1}{\sigma}} \left(\delta_{C_J} C_{tJ} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right] - \frac{\phi}{\phi+1} N_t^{\frac{\phi+1}{\phi}} \right\} \tag{7}$$

In Equation 7, ω_J represents the importance, in the consumer's preferences, of the good produced by industry i at time t. $\delta_{C_J} \in (0,1]$ captures the flow benefit of durables consumption.

The production function is given by:

$$Q_{tJ} = z_{tJ} \left(\frac{K_{t-1,J}}{(1-\mu_J)\,\alpha_J} \right)^{\alpha_J(1-\mu_J)} \left(\frac{N_{tJ}}{(1-\mu_J)(1-\alpha_J)} \right)^{(1-\alpha_J)(1-\mu_J)} \left(\frac{M_{tJ}}{\mu_J} \right)^{\mu_J} \tag{8}$$

$$M_{tJ} = \prod_I \left(\frac{M_{t,I\to J}}{\Gamma_{IJ}^M} \right)^{\Gamma_{IJ}^M} \tag{9}$$

In the production function M_{tJ} denotes total intermediate inputs, which are, in turn, a Cobb-Douglas aggregate of industry-specific inputs. N_{tJ} and K_{tJ} are labor and capital inputs, respectively.

Capital is industry-specific and each $K_{t,J}$ follows the following law of motion:

$$K_{t,J} = X_{tJ} + (1-\delta_K)K_{t-1,J} \tag{10}$$

$$X_{tJ} = \prod_I \left(\frac{X_{t,I\to J}}{\Gamma_{IJ}^X} \right)^{\Gamma_{IJ}^X} \tag{11}$$

Similar to input bundles M_{tJ}, total industry investment consists of a Cobb-Douglas aggregate of industry-specific inputs.

The industry-specific resource constraints are given by:

$$Q_{tJ} = -\left(1-\delta_{C_J}\right) C_{t-1J} + C_{t,J} + \sum_{I=1}^{N} \left[M_{t,J\to I} + X_{t,J\to I} \right] \tag{12}$$

Each industry's output time t output can either be consumed or purchased by the other industries in the economy.

Labor market clearing states that total labor supply equals the total of all labor demanded by the N industries:

$$N_t = \sum_{J=1}^{N} N_{tJ} \tag{13}$$

Last, we need to specify beliefs and the law of motion for the exogenous productivity process. To minimize the distributional we assume that agents have rational expectations and that industry-specific (log) productivity follows a random walk:

$$\mathbb{E}_t[\log z_{t+1,J}] = \log z_{t,J} \tag{14}$$

3.1.2 Equilibrium productivity and employment implications

To solve the model we focus on the social planner's constrained maximization problem: Since this economy satisfies the conditions of the Welfare Theorems, the planner's solution will correspond to an equilibrium outcome. Equations 7 to 14 describe the model.

We solve the model using a first order approximation along the balanced growth path. Based on the first order approximation, we reduce the equilibrium conditions analytically to a system in prices, productivities, and endogenous state variables only. This reduced system is of full rank and can be solved with the standard Blanchard-Kahn algorithm. With durable consumption, the state variables include consumer durables. Without durable consumption, the only endogenous state variables are the industry-specific capital stock. For ease of exposition, we focus on this case in this section. Appendix A provides a detailed derivation and characterization of the equilibrium.

In the case without consumer durables the law of motion for capital is a simple multi-sector version of the standard RBC-model. Specifically:

$$\hat{k}_t \equiv \begin{bmatrix} \hat{k}_{t,1} \\ \vdots \\ \hat{k}_{t,N} \end{bmatrix} = \Pi_{kk} \begin{bmatrix} \hat{k}_{t-1,1} \\ \vdots \\ \hat{k}_{t-1,N} \end{bmatrix} - \Pi_{kz} \begin{bmatrix} \hat{z}_{t,1} \\ \vdots \\ \hat{z}_{t,N} \end{bmatrix} \tag{15}$$

The corresponding policy function for labor supply takes the following form:

$$\hat{N}_t \equiv \begin{bmatrix} \hat{N}_{t,1} \\ \vdots \\ \hat{N}_{t,N} \end{bmatrix} = \Pi_{Lk} \begin{bmatrix} \hat{k}_{t-1,1} \\ \vdots \\ \hat{k}_{t-1,N} \end{bmatrix} - \Pi_{Lz} \begin{bmatrix} \hat{z}_{t,1} \\ \vdots \\ \hat{z}_{t,N} \end{bmatrix} \tag{16}$$

To back out productivity we initialize the economy on the balanced growth path and then back our productivity growth from employment growth in each industry using the following relationship:

$$\Delta\hat{z}_t \equiv \begin{bmatrix} \Delta\hat{z}_{t,1} \\ \vdots \\ \Delta\hat{z}_{t,N} \end{bmatrix} = \Pi_{Lz}^{-1} \begin{bmatrix} \Delta\hat{N}_{t,1} \\ \vdots \\ \Delta\hat{N}_{t,N} \end{bmatrix} - \Pi_{Lz}^{-1}\varrho \begin{bmatrix} \Delta\hat{N}_{t-1,1} \\ \vdots \\ \Delta\hat{N}_{t-1,N} \end{bmatrix} - \Pi_{Lz}^{-1}\Xi \begin{bmatrix} \Delta\hat{z}_{t-1,1} \\ \vdots \\ \Delta\hat{z}_{t-1,N} \end{bmatrix}, \tag{17}$$

where $\Delta\hat{z}_{t,J}$ denotes the log-deviation of productivity growth in industry J from its value along the balanced growth path and similarly for employment growth. The matrices Π_z, ϱ, Ξ are characterized in Appendix A.

Finally, we provide a counterfactual decomposition of aggregate employment due to shocks to industry J by feeding a counterfactual productivity growth series into the system given by (15) and (16), where we set productivity shocks in industries $I \neq J$ to zero. These industry-specific contributions are weighted with the model-implied employment shares along the balanced growth path.

3.1.3 Calibration

We calibrate the technological parameters in our economy using industry-level information from the BEA. For the Γ_{IJ}^M and ω_J parameters, we use data from the post-WWII Input Output tables to obtain the different $\Gamma_{IJ,t}^M$ and $\omega_{J,t}$ and then average over time to get Γ_{IJ}^M and ω_J. At present, we abstract from consumer durables and set $\delta_{C_J} = 1$ for all J. To construct Γ_{IJ}^K, we get a first estimate of Γ_{IJ}^K from the Capital Flows Table.[5] We set the common depreciation rate for capital is set to 2% per quarter. For the factor share α_J, μ_J parameters, we similarly use data from the BEA Industry accounts, taking the ratio of labor compensation and intermediate expenses to gross output, and then average over time.[6]

We explore a range of reasonable preference parameters. Our choice of the quarterly discount rate is standard: $\beta = 1.05^{-1/4}$. We consider macro labor supply elasticity ϕ in $\{1, 3\}$ and values for the intratemporal preference elasticity σ to lie in $\{0.5, 0.75\}$. Note that the latter calibration differs from calibrations of $\sigma > 1$ in the International Trade or New Keynesian literatures that focus on substitution across varieties of the same good. Our two values for σ span the benchmark estimate of from Atalay (2014). Additionally, we consider

[5]Following Foerster, Sarte, and Watson (2011), to account for the substantial maintenance and repair expenditures that McGrattan and Schmitz (1999) report, we add a 25% share to the diagonal entries of the Γ^K matrix.

[6]In future iterations, we should exploit the time variation by allow the above parameters to be slowly, deterministically moving over the sample period.

the Cobb-Douglas calibration of $\sigma = 1$ used in Foerster, Sarte, and Watson (2011).

3.2 Main results

Overall, our model does a good job at replicating aggregate employment dynamics. Because we approximate overall employment growth with a model which features constant employment shares that do not match those in the data exactly, the employment series implied by our model does not reproduce that in the data perfectly. Figure 1 shows the quality of the approximation. While the plot is for the case of $\sigma = 0.75, \phi = 1$, the quality of the approximation is not sensitive to that choice. The correlation of model-implied with actual employment is 0.97, corresponding to an R^2 of about 0.93. The model approximation does, however, overstate the incidence of large shocks somewhat. As we will see, this results in a slightly higher model-implied kurtosis than we observe in the data.

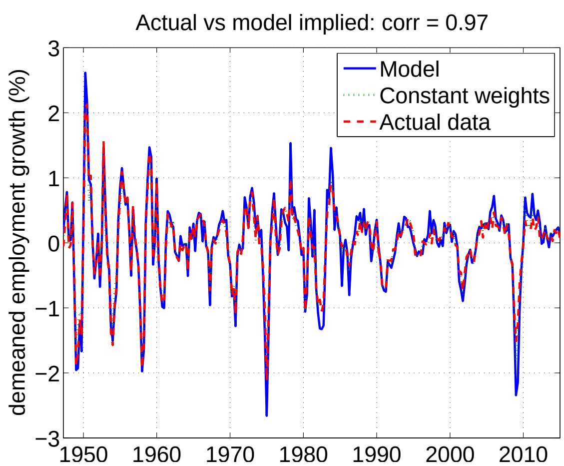

Figure 1: Model-implied employment based on filtered productivty shocks versus actual employment

3.2.1 Implied correlation and IO structure

The two key determinants of aggregate skewness and kurtosis in our model are whether shocks comove and what the sectoral linkages are. Figure 2 visualizes these characteristics for our model economy.

If filtered sectoral productivities were independent, employment implied by productivity growth originating in different sectors would be uncorrelated. The correlation structure in Figure 2(a–c shows, however, that this is not the case: We should expect much of the aggregate kurtosis to stem from correlated industry-specific productivity shocks. The Figure shows that shocks within the secondary and within the tertiary sector tend to be strongly

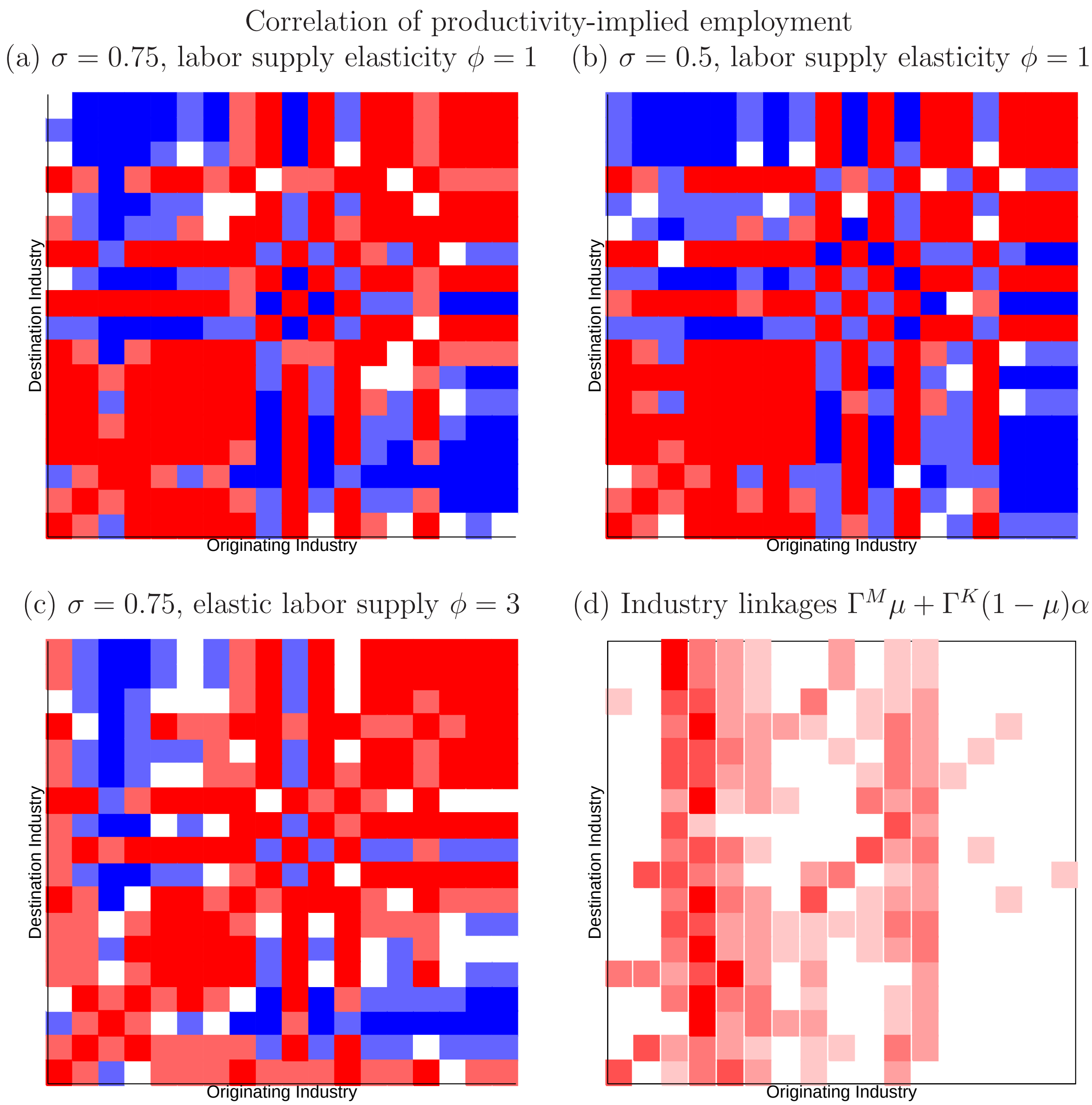

Figure 2: Productivity correlation and strength of IO linkages

Note: Panels (a) through (c) visualize the correlations for five categories: Correlations below -0.4 are shown in dark blue, correlations in $[-0.4, -0.1]$ in light blue, correlations below 0.1 in absolute value in white, and correlations above +0.1 and +0.4 in light and dark red, respectively. Panel (d) shows the linkages for five different categories ranging from below 0.03 (lightest shade of red) to above 0.25 (darkest shade of red).

positive correlated within sector, but negatively correlated with industries outside their own broad sector. Correlations are more positive with a higher intratemporal elasticity of substitution or more elastic labor supply.

How industry-specific productivity shocks propagate depends on the input-output structure of the economy. The input-output structure, Figure 2(d), shows the strongest linkages between construction, both manufacturing sectors, and wholesale as well as between business services and FIRE as originating industries to most downstream industries with the exception of manufacturing and government services. The strength of the input-output linkages, in conjunction with industries' weights in consumers' preferences, determine the Hulten (1978)-type gross sales weights. These weights summarize the importance of industries' productivity shocks.

3.2.2 Skewness and kurtosis

Given the filtered productivity and the employment implied by equations (15) to (17), we can now decompose aggregate employment tail risk that is due to productivity shocks from each industry:[7]

$$\mathbb{S}_i^{tot} = \sum_{j,k,l} \frac{1}{3} \cdot \left(\mathbf{1}_{i=j} + \mathbf{1}_{i=k} + \mathbf{1}_{i=l}\right) \cdot y_j^* y_k^* y_l^* \cdot \frac{\mathbb{E}\left[\Delta \log z_j \Delta \log z_k \Delta \log z_l\right]}{\mathbb{V}^{3/2}} \tag{18}$$

$$\mathbb{K}_i^{tot} = \sum_{j,k,l,m} \frac{1}{4} \left(\mathbf{1}_{i=j} + \mathbf{1}_{i=k} + \mathbf{1}_{i=l} + \mathbf{1}_{i=m}\right) \cdot y_j^* y_k^* y_l^* y_m^* \cdot \frac{\mathbb{E}\left[\Delta \log z_j \Delta \log z_k \Delta \log z_l \Delta \log z_m\right]}{\mathbb{V}^2} - 3\frac{\kappa(i,\Sigma)}{\mathbb{V}^2}, \tag{19}$$

where the constant term $\frac{\kappa(i,\Sigma)}{\mathbb{V}^2}$ sums to one.[8]

Table 3 presents the results of the structural decomposition. Note that skewness and kurtosis measures that assume independent shocks fail dramatically at capturing overall skewness and kurtosis: The implied aggregate skewness is 0.29, compared to -0.53 in the

[7] We assume that the industry shocks are demeaned in what follows.

[8] $\kappa(i, \Sigma)$ is given by

$$\begin{aligned}\kappa(i,\Sigma) &\equiv \Sigma_{i,i}^2 + \sum_j \left(\mathbf{1}_{i\neq j}\Sigma_{i,i}\Sigma_{j,j} + \sum_k \left(\mathbf{1}_{i\neq k}\Sigma_{j,j}\Sigma_{i,k} + \mathbf{1}_{j\neq k}\Sigma_{i,i}\Sigma_{j,k} + \sum_\ell \mathbf{1}_{i\neq j}\mathbf{1}_{k\neq \ell}\Sigma_{i,j}\Sigma_{k,\ell}\right)\right) \\ &= \sum_j \left(\sum_k \left(\sum_\ell \mathbb{V}_{i,j}\mathbb{V}_{k,\ell}\right)\right).\end{aligned} \tag{20}$$

data, and the implied excess kurtosis is 0.34, compared to 2.70 in the data. We therefore focus on the decomposition that allows for cross-sectional dependence in productivity shocks.

Industry	$\mathbb{S}^d$	$\mathbb{S}^{tot}$	$\mathbb{K}^d$	$\mathbb{K}^{tot}$
Agriculture	-0.00	0.01	0.00	0.08
Mining/Petroleum Extraction	0.00	0.00	0.00	0.22
Construction	0.31	-0.55	0.31	2.14
Durable Manufacturing	0.03	-0.01	0.02	1.56
Nondurable Manufacturing	0.00	-0.03	0.00	0.38
Wholesale	0.00	0.03	0.00	0.18
Retail	0.00	-0.01	0.00	0.07
Transport.	0.00	0.01	0.00	0.06
Utilities	-0.00	-0.03	-0.00	-0.04
Information	0.00	-0.01	0.00	0.10
FIRE	-0.02	-0.16	0.00	-0.60
Business Services	0.00	0.00	0.00	0.21
Education/Health	-0.00	0.01	0.00	-0.04
Leisure and Accommodation	-0.00	-0.01	0.00	0.00
Other Services	0.00	0.01	0.00	0.02
Federal Government	-0.00	-0.00	0.00	-0.07
State Government	-0.00	0.02	0.00	-0.45
Local Government	-0.03	0.07	0.01	-1.14
Total	0.29	-0.66	0.34	2.70
Actual (employment)		-0.53		2.70

Table 3: Structural decompositions of employment growth kurtosis, $\sigma = 0.75$

Notes: Due to approximation error, differences between the industry total and actual employment growth may exist. Rescaled employment is scaled so that average industry-wide employment shares correspond to the model-implied shares.

The model-implied skewness and kurtosis overstate that in the data somewhat when allowing for cross-sectional dependence: Total skewness is -0.66, compared to -0.53 in the data, and excess kurtosis is 2.70, equal up to two digits to the kurtosis found in the data. The approximation error is related to the fact that we use constant, model-implied weights. As Figure 1 shows, this leads to counterfactual large employment growth during some episodes.

A few industries are key for explaining aggregate skewness and kurtosis. According to the model, the largest contributor to aggregate skewness in employment growth is the construction industry that explains most of aggregate skewness. FIRE further lowers the skewness, partly offset by the positive skewness of state and local governments. Construction together with durable manufacturing also has the largest absolute contribution to model-implied kurtosis. Also extractive industries and business services contribute to a positive kurtosis. Note that these industries together are also broadly the industries with the largest input-output linkages according to Figure 2(d). State and local governments along with the

FIRE industry are found to lower the overall kurtosis.

To put these results in perspective, Tables 4 and 5 compare the contribution of different industries in the model and in the data. For the baseline calibration of $\sigma = 0.75, \phi = 1$, the observed (negative) skewness is largely explained by construction, as well as FIRE. In the statistical decomposition, in contrast, durable manufacturing is much more important, along with other secondary industries and business services. For the kurtosis, the results of the structural decomposition roughly agree with the statistical decomposition for businesses services and both manufacturing sectors. The statistical decomposition understates, however, the importance of the construction and FIRE industries as well as state and local governments.

Elasticity of substitution σ Elasticity of labor supply ϕ Industry	0.75 1 $\mathbb{S}^{tot}$	0.75 3 $\mathbb{S}^{tot}$	0.5 1 $\mathbb{S}^{tot}$	0.5 3 $\mathbb{S}^{tot}$	1 1 $\mathbb{S}^{tot}$	1 3 $\mathbb{S}^{tot}$	Data $\mathbb{S}^{tot}$
Agriculture	-0.8	-0.4	0.1	0.0	-1.5	-0.8	-2.0
Mining/Petroleum Extraction	-0.2	-0.4	-0.5	-0.7	0.4	-0.1	-0.8
Construction	83.7	3.9	-10.6	-20.3	185.5	38.7	9.4
Durable Manufacturing	2.2	54.8	5.6	58.1	25.9	51.1	40.1
Nondurable Manufacturing	5.0	6.4	3.8	5.7	5.7	6.8	11.3
Wholesale	-4.2	3.1	-0.8	4.7	-4.5	1.6	4.6
Retail	1.7	1.2	1.3	1.3	1.5	0.9	8.0
Transport.	-1.8	0.6	0.1	0.6	-3.9	0.2	5.5
Utilities	4.5	3.3	6.0	3.4	-0.6	2.5	0.5
Information	1.9	2.0	1.5	1.9	2.2	2.1	3.9
FIRE	24.5	20.0	44.6	20.5	-34.5	10.9	3.5
Business Services	-0.7	2.6	1.0	2.3	-2.4	2.5	10.9
Education/Health	-1.5	-0.3	1.4	0.9	-5.3	-2.0	2.3
Leisure and Accommodation	1.4	1.1	2.2	1.5	-0.7	0.2	5.0
Other Services	-1.4	-0.0	-0.4	-0.3	-3.3	-0.1	1.6
Federal Government	0.4	2.9	2.5	2.8	-2.2	2.9	1.6
State Government	-3.7	0.4	12.2	5.2	-17.5	-4.2	-1.9
Local Government	-10.7	-1.2	30.0	12.5	-44.8	-13.2	-3.6
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Actual (employment)	79.9	79.9	81.2	81.2	78.5	78.5	100.0

Table 4: Comparing relative skewness decompositions across calibrations

Comparing different calibrations of the model reveals that the results are sensitive to the calibration. With $\sigma = 0.5$, durable manufacturing and FIRE loses in importance for kurtosis while the opposite is true for $\sigma = 1$. For skewness, FIRE is found to be less important when labor supply is more elastic.

Elasticity of substitution σ	0.75	0.75	0.5	0.5	1	1	
Elasticity of labor supply ϕ	1	3	1	3	1	3	Data
Industry	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$
Agriculture	3.0	2.6	2.5	2.5	3.6	2.8	2.0
Mining/Petroleum Extraction	8.2	10.8	7.7	10.6	8.9	11.4	3.9
Construction	79.4	43.3	85.8	46.1	61.2	32.4	7.9
Durable Manufacturing	57.9	60.4	40.2	52.7	74.7	66.3	45.6
Nondurable Manufacturing	14.2	11.8	13.6	11.5	14.6	12.0	11.9
Wholesale	6.5	7.2	3.2	5.4	9.7	8.8	2.4
Retail	2.7	1.7	2.7	1.8	2.1	1.3	4.4
Transport.	2.2	2.4	1.8	2.2	2.9	2.8	5.2
Utilities	-1.4	-1.6	0.4	-0.4	-2.9	-2.4	1.0
Information	3.6	3.5	3.5	3.5	3.7	3.6	4.4
FIRE	-22.1	-15.7	-4.6	-5.5	-34.7	-21.1	1.0
Business Services	8.0	7.6	6.6	6.7	9.8	9.0	4.3
Education/Health	-1.4	-1.0	-0.9	-0.8	-1.7	-1.0	1.6
Leisure and Accommodation	0.1	-0.1	0.6	0.2	-0.5	-0.3	2.3
Other Services	0.9	0.2	0.9	0.2	1.2	0.5	0.6
Federal Government	-2.6	-1.8	-2.2	-1.7	-2.9	-1.9	1.4
State Government	-16.7	-8.6	-17.5	-9.8	-14.2	-6.6	-0.2
Local Government	-42.4	-22.9	-44.3	-25.4	-35.6	-17.6	0.4
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Actual (employment)	99.8	99.8	99.6	99.6	100.2	100.2	100.0

Table 5: Comparing relative kurtosis decompositions across calibrations

3.3 Role of preferences

Our results are clearly sensitive to the calibration of the preference parameters governing the elasticity of substitution across industries and the labor supply elasticity.

The main effect of increasing the elasticity of labor supply is to scale down the implied size of sectoral shocks, as can be seen by comparing the magnitudes of the model-implied productivity shocks in Figure 5 in the Appendix: Comparing the panels on the left with $\phi = 1$ to the ones on the right with $\phi = 3$ shows that the shape of the shocks are similar, but the scale of the shocks is reduced by roughly one quarter when moving from the low to the high labor supply elasticity. Intuitively, wages are a key determinant of marginal costs in our model and with more elastic labor supply given movements in employment imply smaller movements in the wage – and hence smaller productivity shocks to yield the same relative price pattern across industries that explains the equilibrium allocation. This can be seen in a static version of our model without wealth effects on labor supply and capital as in Carvalho and Gabaix (2013). In this model, we have a closed form expression for productivity growth that is given by:

$$\begin{aligned}\Delta \log z_{it} = &\left(1 + \frac{1-\sigma}{\phi}\right) \frac{1-\mu_i}{1-\sigma} \cdot \left[\frac{\sum_k y^*_{kt} \Delta \log N_{kt} - \sum_k \sum_l \mu_l \Gamma_{kl,t} y^*_{lt} \Delta \log N_{lt}}{\sum_k y^*_k - \sum_k \sum_l \mu_l \Gamma_{kl,t} y^*_{lt}}\right] \\ &- \frac{1}{1-\sigma} \sum_j \left(I - (\mathrm{diag}(\mu).\Gamma^M)'\right)_{jit} \left[\frac{y^*_{jt} \Delta \log N_{jt} - \sum_l \mu_l \Gamma_{jlt} y^*_{lt} \Delta \log N_{lt}}{y^*_{jt} - \sum_l \mu_l \Gamma_{jlt} y^*_{lt}}\right]\end{aligned} \tag{21}$$

Clearly, a higher labor supply elasticity reduces productivity growth across industries for given employment growth. The size of this effects affects only the first component of productivity growth and does depend on the labor share in each industry, $1 - \mu_i$. This may explain why the relative shock contributions to skewness and kurtosis discussed previously are not invariant to the labor supply elasticity.

The elasticity of substitution across industries also affects the magnitude of the inferred productivity shocks, as well as the persistence of their deviations from mean. A first order effect of making goods more complementary by lowering σ is also that shocks are found to be smaller in magnitude. With more elastic industry demands, our model also infers bigger and slower swings in productivity growth; see Figure 5 in the Appendix. We take the latter as tentative evidence in favor of the model with a lower elasticity of substitution.

3.4 Role of common factors

It is possible that the shocks to industry productivity we identify are largely driven by a single common productivity shock. We now allow for this possibility and define a new "aggregate" productivity shock as the average productivity growth shock across all industries in a given quarter. In other words, we define the aggregate shock as the time fixed effects of the industry-specific productivity shocks. In keeping with our model, we append to the capital and employment policy functions (15) and (16) a column vector of with the sum of the exposure to the industry-specific shocks. This changes only the accounting, but not the aggregate implications. The results of this exercise are presented in Tables 6 and 7.

It is worth noting that the average industry-specific shock explains between roughly one third and two thirds of aggregate employment growth or less, as measured by the R^2, over our sample period, depending on preference parameters (see Figures 4 in the Appendix).[9] The larger these aggregate contributions, the better the aggregate shock also seems to be able to explain higher moments: The common aggregate shock explains about 60-100% of total kurtosis, and 80-130% of observed total skewness – and less so when labor supply is more elastic. This also applies to specific historic episodes such as the most recent recession: For example in the case of $\sigma = 0.75$ and elastic labor supply $\phi = 3$ the common productivity factor explains about 4% of the total cumulative employment decline of 6%, but Construction and Durables Manufacturing have contributions that are almost as important.

The aggregate average productivity shock does not, however, eliminate the industry-specific contributions to kurtosis and skewness: For skewness individual industries explain between -70% to +70% of skewness as well even when aggregate shocks explain 130% of model-implied skewness in the $\sigma = 0.75, \phi = 1$ case. The decomposition results for decomposed kurtosis are not quite as dramatic, but also here sizable industry specific components remain. The overexplaining of aggregate kurtosis by individual components is less of an issue when labor supply is elastic, but in the case the aggregate common factor also has less explanatory power. Together, these results are indicative of an actual role of industry-specific shocks that are not captured by a single aggregate component.

4 Next Steps

The work presented in this draft is highly preliminary. There are at least five directions along which the analysis presented in Sections 2 and 3 should be extended, in addition to robustness checks.

The proposed model extensions are:

- *Consumption durability and adjustment costs.* Our current calibration identifies secondary sectors such as durable manufacturing and construction as pivotal sectors, but

[9]Note that this result is in line with the estimates in Atalay (2014).

Elasticity of substitution σ	0.75	0.75	0.5	0.5	1	1	
Elasticity of labor supply ϕ	1	3	1	3	1	3	Data
Industry	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$	$\mathbb{S}^{tot}$
Agriculture	-4.6	-2.3	-2.0	-1.3	-6.8	-3.5	-2.7
Mining/Petroleum Extraction	-3.3	-2.2	-2.4	-2.0	-3.5	-2.4	-1.5
Construction	73.8	2.7	-5.1	-14.7	158.4	29.1	5.7
Durable Manufacturing	-71.9	7.5	-57.0	15.4	-42.2	2.0	31.1
Nondurable Manufacturing	-1.5	5.6	4.8	6.9	-8.0	3.8	6.5
Wholesale	-9.6	0.0	-4.4	2.3	-10.7	-2.1	1.1
Retail	-0.6	-0.1	-0.5	0.0	-0.5	-0.3	-0.1
Transport.	-3.1	0.6	1.4	1.7	-7.6	-0.8	3.0
Utilities	1.6	1.1	2.4	0.7	-1.7	1.0	0.0
Information	-0.7	0.6	-0.2	0.7	-0.9	0.4	2.0
FIRE	13.4	10.8	30.3	10.2	-37.1	4.3	-0.7
Business Services	-4.8	1.1	0.0	2.0	-9.4	-0.4	3.7
Education/Health	-0.7	-0.3	0.1	-0.4	-1.9	-0.5	-4.9
Leisure and Accommodation	0.2	0.1	0.5	0.2	-0.7	-0.2	-1.0
Other Services	0.4	2.1	3.8	2.8	-4.4	0.9	-0.8
Federal Government	0.8	2.6	1.0	1.4	0.4	3.8	-0.7
State Government	-5.3	-3.6	0.2	-4.6	-10.4	-3.5	-4.7
Local Government	-14.4	-11.1	0.3	-11.7	-27.0	-11.4	-11.5
Aggregate	130.5	85.0	126.6	90.5	114.0	80.0	75.6
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Actual (employment)	79.9	79.9	81.2	81.2	78.5	78.5	100.0

Table 6: Comparing skewness decompositions

Elasticity of substitution σ	0.75	0.75	0.5	0.5	1	1	
Elasticity of labor supply ϕ	1	3	1	3	1	3	Data
Industry	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$	$\mathbb{K}^{tot}$
Agriculture	-2.8	-1.9	-2.9	-1.7	-2.4	-1.9	-1.6
Mining/Petroleum Extraction	4.5	7.8	4.0	7.6	5.4	8.4	2.4
Construction	43.2	14.2	52.5	19.1	23.1	1.2	3.8
Durable Manufacturing	16.2	26.1	-10.7	12.6	44.9	38.8	32.5
Nondurable Manufacturing	-17.1	-11.4	-15.7	-10.5	-16.8	-11.6	1.8
Wholesale	0.5	2.3	-3.0	0.5	4.1	4.1	-1.5
Retail	-0.6	-0.6	-1.0	-0.8	-0.4	-0.4	-3.6
Transport.	-4.8	-2.9	-4.9	-2.9	-3.9	-2.4	1.3
Utilities	-1.4	-1.5	-0.4	-0.9	-2.0	-1.6	0.2
Information	0.2	0.9	0.1	0.9	0.5	1.0	1.7
FIRE	-17.5	-11.7	-5.2	-5.2	-24.9	-13.1	-2.3
Business Services	-1.4	0.2	-2.4	-0.4	0.3	1.2	-0.9
Education/Health	0.5	1.0	0.3	0.7	0.9	1.6	-2.3
Leisure and Accommodation	-0.0	0.0	0.0	-0.0	0.0	0.2	-2.6
Other Services	-4.7	-4.1	-3.9	-3.5	-4.6	-4.0	-0.9
Federal Government	0.0	0.6	-0.1	0.4	0.2	0.8	-2.1
State Government	-1.4	4.0	-2.0	3.5	-0.3	4.8	-2.1
Local Government	-4.3	8.6	-5.4	7.8	-1.3	10.4	-5.5
Aggregate	90.9	68.3	100.5	72.9	77.1	62.6	81.6
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Actual (employment)	99.8	99.8	99.6	99.6	100.2	100.2	100.0

Table 7: Comparing kurtosis decompositions allowing for an aggregate shock

their precise contributions are sensitive to the calibration of preference parameters. Calibrating the model to allow for consumption durability and adjustment costs in the investment rate $\frac{X_{tJ}}{K_{t-1,J}}$ can help pin down the implied productivity contributions.

- *Time-varying Input-Output parameters.* We can allow for exogenous time-variation in factor shares and input-output linkages. Starting from our baseline model, this extension will allow us to assess whether the documented tail behavior can be attributed to economic fundamentals via changing input-output linkages, thereby connecting to the literature on stochastic volatility. Incorporating this extension into the dynamic model requires modeling agents' expectations about parameter changes.

- *Initial conditions:* Currently we initialize our economic filter at the steady state. An alternative possibility is to draw an initial condition from the model-implied long-run distribution of states. Since the latter will depend on the initial conditions, this extension would require us to iterate until the long-run distribution stabilizes.

- *Prices:* It would be interesting to use information on prices to discipline our model and to distinguish shocks to demand from preference shocks.

- Finally, there are *alternative decompositions* that we could compute to characterize the contribution of each industry to aggregate skewness and kurtosis. Using an alternate decomposition, Barnichon (2012) computes the contribution of vacancy postings and job separations to skewness and kurtosis of unemployment.

Robustness checks include:

- *Finer industry definitions:* The Bureau of Economic Analysis (BEA) has data on value added, by industry, for each year between 1947 and the present. The industry-level classification used by the BEA is roughly the 2-digit level, which is substantially more detailed than that of the BLS Current Employment Survey data.

- *Differences across periods*: Tail risk of non-farm employment growth rates has become more pronounced over time. Skewness in aggregate quarterly employment growth was -0.56 in the first half of the sample (1947-1979), and -1.26 in the second half of the sample. Excess kurtosis is was 1.09 and 2.23 in the two parts of the sample. These differences reflect in part, but decidedly not in whole, the Great Recession. In a future draft, we plan on computing the fraction of these differences that are due to industry composition, as opposed to differences in individual industries' tail risks.

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A Deriving the Extended Model Filter

A.1 The Model

The general model takes the following form:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log \left[\bar{C}_t^{\tilde{b}} \left[\sum_{J=1}^{N} \omega_J^{\frac{1}{\sigma}} \left(\delta_{C_J} C_{tJ}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right] - \frac{\phi}{\phi+1} \left(\sum_{J=1}^{N} N_{tJ} \right)^{\frac{\phi+1}{\phi}} \right\}$$

where $\bar{C}_t$ is given and $\tilde{b} = \frac{\sigma-1}{\sigma} b$. Note that for $b = 1$ so that $\tilde{b} = \frac{\sigma-1}{\sigma}$ and with $\bar{C}_t = C_t$ in equilibrium, the equilibrium conditions collapse to those of a simple economy without wealth effects on labor supply.

Maximization is subject to the following constraints:

$$Q_{tJ} = -\left(1-\delta_{C_J}\right) C_{t-1J} + C_{t,J} + \sum_{I=1}^{N} \left[M_{t,J\to I} + X_{t,J\to I} \right]$$

$$K_{t,J} = X_{tJ} + (1-\delta_K) K_{t-1,J} \tag{22}$$

$$X_{tJ} = \prod_I \left(\frac{X_{t,I\to J}}{\Gamma_{IJ}^X} \right)^{\Gamma_{IJ}^X} \tag{23}$$

$$M_{tJ} = \prod_I \left(\frac{M_{t,I\to J}}{\Gamma_{IJ}^M} \right)^{\Gamma_{IJ}^M} \tag{24}$$

$$Q_{tJ} = z_{tJ} \left(\frac{K_{t-1,J}}{\left(1-\mu_J\right)\alpha_J} \right)^{\alpha_J(1-\mu_J)} \left(\frac{N_{tJ}}{\left(1-\mu_J\right)\left(1-\alpha_J\right)} \right)^{(1-\alpha_J)(1-\mu_J)} \left(\frac{M_{tJ}}{\mu_J} \right)^{\mu_J} \tag{25}$$

We focus again on the social planner's constrained maximization problem: Since this economy satisfies the conditions of the Welfare Theorems, the planner's solution will correspond to an equilibrium outcome. In this constrained maximization problem, let P_{tJ} refer to the Lagrange multiplier associated with the market-clearing-condition for the output of industry J in period t, and let P_{tJ}^{inv} refer to the Lagrange multiplier associated with the market-clearing condition for the industry J investment good in period t. Finally, $\mathbb{E}_t$ refers to the expectation operator; the expectations are formed at time t.

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log \left[\bar{C}_t^{\tilde{b}} \left[\sum_{J=1}^{N} \omega_J^{\frac{1}{\sigma}} \left(\delta_{C_J} C_{tJ}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right] - \frac{\phi}{\phi+1} \left(\sum_{J=1}^{N} N_{tJ} \right)^{\frac{\phi+1}{\phi}} \right\}$$
$$- \frac{\phi}{\phi+1} \left(\sum_{J=1}^{N} N_{tJ} \right)^{\frac{\phi+1}{\phi}} + \sum_{J=1}^{N} P_{tJ}^{inv} \left[X_{tJ} + (1-\delta_K) K_{t-1,J} - K_{t,J} \right]$$

$$+\sum_{J=1}^{N} P_{tJ}\left[Q_{tJ}+\left(1-\delta_{C_J}\right)C_{t-1,J}-C_{tJ}-\sum_{I=1}^{N}\left[M_{t,J\to I}+X_{t,J\to I}\right]\right]\Bigg\} \quad .$$

The first-order necessary conditions are:

$$[C_{tJ}]: P_{tJ}=\left(\omega_J\right)^{\frac{1}{\sigma}}\left(\delta_{C_J}\right)^{\frac{\sigma-1}{\sigma}}\left(C_{tJ}\right)^{-\frac{1}{\sigma}}\bar{C}_t^{\tilde{b}}\left(\sum_{I=1}^{N}\left(\omega_I\right)^{\frac{1}{\sigma}}\left(\delta_{C_I}\cdot C_{tI}\right)^{\frac{\sigma-1}{\sigma}}\right)^{-1}$$
$$+\beta\mathbb{E}_t[P_{t+1,J}]\left(1-\delta_{C_J}\right) \quad . \tag{26}$$

$$[M_{t,I\to J}]: \frac{Q_{tJ}\mu_J\Gamma_{IJ}^{M}}{M_{t,I\to J}}=\frac{P_{tI}}{P_{tJ}} \quad . \tag{27}$$

$$[X_{t,I\to J}]: P_{tI}=P_{tJ}^{inv}\frac{X_{tJ}\cdot\Gamma_{IJ}^{X}}{X_{t,I\to J}} \quad . \tag{28}$$

$$[N_{tJ}]: \left(\sum_{J'=1}^{N}N_{tJ'}\right)^{\frac{1}{\phi}}=\frac{\left(1-\alpha_J\right)\left(1-\mu_J\right)P_{tJ}Q_{tJ}}{N_{tJ}} \tag{29}$$

$$[K_{t,J}]: P_{tJ}^{inv}=\beta\cdot\mathbb{E}_t\left[\frac{P_{t+1,J}Q_{t+1,J}\left(1-\mu_J\right)\alpha_J}{K_{t,J}}\right]+\beta(1-\delta_K)\mathbb{E}_t\left[P_{t+1,J}^{inv}\right] \quad . \tag{30}$$

Re-stating the market-clearing condition of each industry J:

$$Q_{tJ}=C_{tJ}-\left(1-\delta_{C,J}\right)C_{t-1,J}+\sum_{I=1}^{N}\left[X_{t,J\to I}+M_{t,J\to I}\right] \quad . \tag{31}$$

A.2 Finding the Steady State

In this section we derive the steady state consumption, materials, and investment shares as well as industry-level employment shares. To do this, we assume that productivity growth has no deterministic trend component nor drift. Alternatively, with trending or drifting productivity we would express the steady state as relative to trend along a balanced growth. We focus on the case of $b=0$.

The first-order necessary conditions are (just dropping the time subscripts from the Lagrangian's FOC and then re-arranging, cancelling terms when possible):

$$\left(P_J\right)^{\sigma}=\frac{\left(\delta_{C_J}\right)^{\sigma}}{\left[1-\beta\left(1-\delta_{C_J}\right)\right]^{\sigma}}\omega_J\left(\delta_{C_J}C_J\right)^{-1}\bar{C}^{1-\sigma+b-\frac{1}{\sigma}b}.$$

$$\frac{Q_J\mu_J\Gamma_{IJ}^{M}}{M_{I\to J}}=\frac{P_I}{P_J} \quad .$$

$$P_I=P_J^{inv}\frac{\delta_K K_J\Gamma_{IJ}^{X}}{X_{I\to J}} \quad .$$

$$\left(\sum_{J'=1}^{N} N_{J'}\right)^{\frac{1}{\phi}} = \frac{(1-\alpha_J)(1-\mu_J) P_J Q_J}{N_J}$$

$$(1-\beta(1-\delta_K)) P_J^{inv} = \beta \cdot \frac{P_J Q_J (1-\mu_J)\alpha_J}{K_J}.$$

$$\delta_K K_J = X_J$$

Plug in the cost-minimization-related FOC into the production function:

$$Q_J = z_J \left(\frac{K_J}{\alpha_J(1-\mu_J)}\right)^{\alpha_J(1-\mu_J)} \left(\frac{N_J}{(1-\alpha_J)(1-\mu_J)}\right)^{(1-\alpha_J)(1-\mu_J)} \left(\frac{M_J}{\mu_J}\right)^{\mu_J}$$

$$Q_J = z_J \left(\frac{\beta P_J Q_J}{(1-\beta(1-\delta_K)) P_J^{inv}}\right)^{\alpha_J(1-\mu_J)} \left(\frac{P_J Q_J}{W}\right)^{(1-\alpha_J)(1-\mu_J)} \prod_I \left(\frac{P_J Q_J}{P_I}\right)^{\Gamma_{IJ}^M \mu_J}$$

$$P_J^{-1} = z_J \left(\frac{\beta}{(1-\beta(1-\delta_K))}\right)^{\alpha_J(1-\mu_J)} \left(\frac{1}{W}\right)^{(1-\alpha_J)(1-\mu_J)} \prod_I \left(\frac{1}{P_I}\right)^{\Gamma_{IJ}^M \mu_J + \Gamma_{IJ}^X (1-\mu_J)\alpha_J}$$

$$P_J = (z_J)^{-1} \left(\frac{(1-\beta(1-\delta_K))}{\beta}\right)^{\alpha_J(1-\mu_J)} W^{(1-\alpha_J)(1-\mu_J)} \prod_I (P_I)^{\Gamma_{IJ}^M \mu_J + \Gamma_{IJ}^X (1-\mu_J)\alpha_J}$$

$$\log P_J = -\log z_J + \alpha_J(1-\mu_J)\log\left[\frac{1-\beta(1-\delta_K)}{\beta}\right] + (1-\alpha_J)(1-\mu_J)\log W + \sum_I \underbrace{\Gamma_{IJ}^M \mu_J + \Gamma_{IJ}^X (1-\mu_J)\alpha_J}_{\Gamma_{IJ}} \log P_I$$

In what follows we approximate around a constant $\log z_J$. Alternatively, this can be justified by interpreting $z_{J,t}$ as deviations from a common trending or drifting term $\bar{z}_t$ and expressing prices in quantities relative to trend.[10]

Write the last equation in matrix form:

$$\log P = (I - \Gamma')^{-1}\left[(1-\mu)\circ\left[\alpha\log\left[\frac{1-\beta(1-\delta_K)}{\beta}\right] + (1-\alpha)\log W\right]\right]$$

Setting the wage to be the numeraire good to get:

$$\log P = (I - \Gamma')^{-1}\left[(1-\mu)\circ\alpha\log\left[\frac{1-\beta(1-\delta_K)}{\beta}\right]\right]$$

[10]In the presence of a common trending or drifting term and with common factor shares quantities $Q_{J,t}$ rise at rate z^{γ} with $\gamma = (1+\mu+(1-\mu)\alpha)^{-1}$ and prices $P_{J,t}, P_{J,t}^{inv}$ fall at the same rate along a balanced growth path.

Use the market clearing condition for good J (again letting $Y_J = Q_J P_J$):

$$
\begin{aligned}
Y_J &= P_J \delta_{C_J} C_J + \sum_{I=1}^{N} \left[P_J X_{J \to I} + P_J M_{J \to I} \right] \\
Y_J &= \frac{(\delta_{C_J})^{\sigma}}{\left[1 - \beta \left(1 - \delta_{C_J}\right)\right]^{\sigma}} \omega_J \left(P_J \bar{C} \right)^{1-\sigma} + \sum_{I=1}^{N} \underbrace{\left[\frac{\left(1 - \mu_I\right) \alpha_I \beta \delta_K}{\left(1 - \beta(1 - \delta_K)\right)} \Gamma^X_{JI} + \mu_I \Gamma^M_{JI} \right]}_{\tilde{\Gamma}_{JI}} Y_I \\
Q &= \left[\left(I - \tilde{\Gamma} \right)^{-1} \cdot \left(\omega \circ P^{1-\sigma} \circ \frac{(\delta_C)^{\sigma}}{\left[1 - \beta \left(1 - \delta_C\right)\right]^{\sigma}} \right) \right] \cdot P^{-1} \circ \bar{C}^{1-\sigma} \\
\delta_C \cdot C &= \omega \circ P^{-\sigma} \circ \frac{(\delta_C)^{\sigma}}{\left[1 - \beta \left(1 - \delta_C\right)\right]^{\sigma}} \bar{C}^{1-\sigma} \\
L &= \alpha \circ \left[\left(I - \tilde{\Gamma} \right)^{-1} \cdot \left(\omega \circ P^{1-\sigma} \right) \right]
\end{aligned}
$$

Now we can write out the steady-state fractions:

$$
\begin{aligned}
S^L_L &= \mathbf{1} \frac{L'}{\|L\|_1} \\
\tilde{S}^Q_C &= \delta_C \cdot \omega \circ P^{1-\sigma} \circ \frac{(\delta_C)^{\sigma}}{\left[1 - \beta \left(1 - \delta_C\right)\right]^{\sigma}} \circ \left[\left(I - \tilde{\Gamma} \right)^{-1} \cdot \left(\omega \circ P^{1-\sigma} \right) \right]^{-1}
\end{aligned}
$$

$$
\frac{M_{J \to I}}{Q_J} = \frac{Q_I P_I}{Q_J P_J} \cdot \mu_I \Gamma_{JI} = \frac{Y_I}{Y_J} \cdot \mu_I \Gamma_{JI}
$$

Thus

$$
\log S^Q_{M, J \to I} = \log Y_I - \log Y_J + \log \left[\mu_I \Gamma^M_{JI} \right]
$$

And Similarly

$$
\log S^Q_{X, J \to I} = \log Y_I - \log Y_J + \log \left[\frac{\left(1 - \mu_I\right) \alpha_I \beta \delta_K}{\left(1 - \beta \left(1 - \delta_K\right)\right)} \right]
$$

For future reference, define the following matrices:

$$
\begin{aligned}
S_K &= \operatorname{diag}(\alpha \circ (1 - \mu)) \\
S_L &= \operatorname{diag}\left((1 - \alpha) \circ (1 - \mu) \right) \\
S_M &= \operatorname{diag}(\mu) \\
\tilde{S}^Q_C &= \operatorname{diag}(S^Q_C)
\end{aligned}
$$

$$
\tilde{S}^M = \begin{pmatrix}
\Gamma^M_{11} & & & & \Gamma^M_{21} & & & & \dots & \Gamma^M_{N1} & & & \\
& \Gamma^M_{12} & & & & \Gamma^M_{22} & & & \dots & & \Gamma^M_{N2} & & \\
& & \dots & & & & \dots & & \dots & & & \dots & \\
& & & \Gamma^N_{1N} & & & & \Gamma^M_{2N} & .. & & & & \Gamma^M_{NN}
\end{pmatrix}
$$

$$
\tilde{S^Q_X} = \begin{pmatrix}
S^Q_{X,11} & S^Q_{X,12} & \dots & S^Q_{X,1N} & 0 & \dots & & & & & \\
& & \dots & 0 & S^Q_{X,21} & & \dots & & S^Q_{X,2N} & 0 & \dots & \\
& & \dots & & & & \dots & \dots & & & \dots & \\
& & & & & & & \dots & 0 & S^Q_{X,N1} & \dots & S^Q_{X,NN}
\end{pmatrix}
$$

$$
\tilde{S}^Q_M = \begin{pmatrix}
S^Q_{M,11} & S^Q_{M,12} & \dots & S^Q_{M,1N} & 0 & \dots & & & & & \\
& & \dots & 0 & S^Q_{M,21} & & \dots & & S^Q_{M,2N} & 0 & \dots & \\
& & \dots & & & & \dots & \dots & & & \dots & \\
& & & & & & & \dots & 0 & S^Q_{M,N1} & \dots & S^Q_{M,NN}
\end{pmatrix}
$$

$$
S^X_1 = \begin{pmatrix}
\Gamma^X_{11} & & & & \Gamma^X_{21} & & & & \dots & \Gamma^X_{N1} & & & \\
& \Gamma^X_{12} & & & & \Gamma^X_{22} & & & \dots & & \Gamma^X_{N2} & & \\
& & \dots & & & & \dots & & \dots & & & \dots & \\
& & & \Gamma^X_{1N} & & & & \Gamma^X_{2N} & .. & & & & \Gamma^X_{NN}
\end{pmatrix}
$$

A.3 Log-Linearized Equations

We now approximate the model around the above (detrended) steady state. The "hat" notation stands for percentage deviation from steady state in what follows.

$$
\begin{aligned}
\hat{K}_{t,J} &= \delta_K \hat{X}_{tJ} + (1-\delta_K)\hat{K}_{t-1,J} \\
\hat{X}_{tJ} &= \sum_I S^X_{IJ} \cdot \hat{X}_{t,I\to J} \\
\hat{M}_{tJ} &= \sum_I S^M_{IJ} \cdot \hat{M}_{t,I\to J}
\end{aligned}
$$

In the previous equations, S^X_{IJ} and S^M_{IJ} are the steady-state fraction of capital (materials) expenditures of industry J that are purchased from industry I.

$$
\begin{aligned}
\hat{Q}_{tJ} - \hat{M}_{t,I\to J} &= \hat{P}_{tI} - \hat{P}_{tJ} \\
\hat{X}_{tJ} - \hat{X}_{t,I\to J} &= \hat{P}_{tI} - \hat{P}^{inv}_{tJ} \\
\hat{P}_{tJ} + \hat{Q}_{tJ} - \hat{L}_{tJ} &= \frac{1}{\phi}\sum_{I=1}^{N} S^L_I \hat{L}_{tI}
\end{aligned}
$$

In the previous equation, S_I^L is the steady state fraction of labor that is employed in industry I.

$$\hat{P}_{tJ}^{inv} = \beta(1-\delta_K)\hat{P}_{t+1,J}^{inv} + (1-\beta(1-\delta_K))\left[\hat{P}_{t+1,J} + \hat{Q}_{t+1,J} - \hat{K}_{t+1,J}\right]$$

$$\hat{Q}_{tJ} = \frac{1}{\delta_{C_J}} S_{C_J}^Q \hat{C}_{t,J} - S_C^Q \frac{1-\delta_{C_J}}{\delta_{C_J}} \hat{C}_{t-1,J} + \sum_{I=1}^{N}\left[S_{X_{J\to I}}^Q \hat{X}_{t,J\to I} + S_{M_{J\to I}}^Q \hat{M}_{t,J\to I}\right]$$

$S_{\zeta_J}^Q$ is the steady-state fraction of good J that is used for the purpose of $\zeta \in \{C_J,\ X_{J\to I},\ M_{J\to I}\}$.

$$\hat{P}_{tJ} = (1-\delta_{C_J})\,\hat{P}_{t+1,J} - \frac{\delta_{C_J}}{\sigma}\hat{C}_{t,J} - \delta_{C_J}\frac{\sigma-1}{\sigma}\sum_{I=1}^{N}(1-b)S_I^C \hat{C}_{t,I}$$

S_I^C is the steady state fraction of consumption expenditures that comes from good I.

A.4 System Reduction

Note: $\hat{k}$, $\hat{\mathbb{X}}_t$, $\hat{\mathbb{M}}_t$, $\hat{p}_t^{inv}$, $\hat{p}_t$, $\hat{q}_t$, and $\hat{c}$ are $N\times 1$ vectors. $\hat{x}_t$ and $\hat{m}_t$ are of dimension $N^2\times 1$. $T_1 \equiv 1_{N\times 1}\otimes I$ and $T_2 \equiv I \otimes 1_{N\times 1}$.
Initial system of equations:

$$\begin{aligned}
\hat{k}_t &= \delta_K \hat{\mathbb{X}}_t + (1-\delta_K)\,\hat{k}_{t-1} \\
\hat{x}_t &= T_1\hat{\mathbb{X}}_t + T_1\hat{p}_t^{inv} - T_2\hat{p}_t \\
\hat{m}_t &= T_1\hat{q}_t + [T_1 - T_2]\,\hat{p}_t \\
\left(\frac{1}{\phi}S_L^L + I\right)\hat{l}_t &= \hat{p}_t + \hat{q}_t \\
\hat{p}_t^{inv} &= \beta(1-\delta_K)\mathbb{E}_t[\hat{p}_{t+1}^{inv}] + (1-\beta(1-\delta_K))\left[\mathbb{E}_t[\hat{p}_{t+1} + \hat{q}_{t+1}] - \hat{k}_t\right] \\
\hat{q}_t &= (\delta_C)^{-1}\tilde{S}_C^Q \hat{c}_t - \left((\delta_C)^{-1} - I\right)\tilde{S}_C^Q \hat{c}_{t-1} + \tilde{S_X^Q}\hat{x}_t + \tilde{S}_M^Q \hat{m}_t \\
\hat{q}_t &= \hat{z}_t + S_K\hat{k}_{t-1} + S_L\hat{l}_t + S_M\hat{\mathbb{M}}_t \\
\hat{p}_t &= \beta\,(I-\delta_C)\,\hat{p}_{t+1} - \frac{1}{\sigma}\,(I - \beta\,(I-\delta_C))\left[I + S_I^C\,(\sigma-1)\right]\hat{c}_t
\end{aligned}$$

Step 1: Substitute out $\hat{x}_t$ and $\hat{m}_t$

$$\hat{k}_t = \delta_K \hat{\mathbb{X}}_t + (1-\delta_K)\,\hat{k}_{t-1}$$

$$\left(\frac{1}{\phi}S_L^L+I\right)\hat{l}_t = \hat{p}_t + \hat{q}_t$$

$$\hat{p}_t^{inv} = \beta(1-\delta_K)\mathbb{E}_t[\hat{p}_{t+1}^{inv}] + (1-\beta(1-\delta_K))\left[\mathbb{E}_t[\hat{p}_{t+1}+\hat{q}_{t+1}] - \hat{k}_t\right]$$

$$\left[I - \tilde{S}_M^Q T_1\right]\hat{q}_t = (\delta_C)^{-1}\tilde{S}_C^Q\hat{c}_t - \left((\delta_C)^{-1} - I\right)\tilde{S}_C^Q\hat{c}_{t-1} + \tilde{S_X^Q}T_1\hat{\mathbb{X}}_t + \tilde{S}_X^Q T_1\hat{p}_t^{inv}$$

$$+ \left[\tilde{S}_M^Q T_1 - \tilde{S}_M^Q T_2 - \tilde{S_X^Q}T_2\right]\hat{p}_t$$

$$\hat{q}_t = \hat{z}_t + S_K\hat{k}_{t-1} + S_L\hat{l}_t + S_M\hat{\mathbb{M}}_t$$

$$\hat{p}_t = \beta\left(I-\delta_C\right)\hat{p}_{t+1} - \frac{1}{\sigma}\left(I - \beta\left(I-\delta_C\right)\right)\left[I + S_I^C\left(\sigma-1\right)\right]\hat{c}_t$$

Step 2: Re-write the investment Euler equation using that $S_1^X\hat{p}_t = \hat{p}_t^{inv}$ from pre-multiplying the no-arbitrage equation for investment inputs with S_1^X:

$$\hat{k}_t = \delta_K\hat{\mathbb{X}}_t + \left(1-\delta_K\right)\hat{k}_{t-1}$$

$$\hat{x}_t = T_1\hat{\mathbb{X}}_t + T_1\hat{p}_t^{inv} - T_2\hat{p}_t$$

$$\hat{m}_t = T_1\hat{q}_t + \left[T_1 - T_2\right]\hat{p}_t$$

$$\left(\frac{1}{\phi}S_L^L+I\right)\hat{l}_t = \hat{p}_t + \hat{q}_t$$

$$S_1^X\hat{p}_t = \left[S_1^X\beta(1-\delta_K) + \tilde{\beta}\right]\mathbb{E}_t[\hat{p}_{t+1}] + \tilde{\beta}\left[\mathbb{E}_t[\hat{q}_{t+1}] - \hat{k}_t\right]$$

$$\left[I - \tilde{S}_M^Q T_1\right]\hat{q}_t = (\delta_C)^{-1}\tilde{S}_C^Q\hat{c}_t - \left((\delta_C)^{-1} - I\right)\tilde{S}_C^Q\hat{c}_{t-1} + \tilde{S_X^Q}T_1\hat{\mathbb{X}}_t + \tilde{S}_X^Q T_1 S_1^X\hat{p}_t$$

$$+ \left[\tilde{S}_M^Q T_1 - \tilde{S}_M^Q T_2 - \tilde{S_X^Q}T_2\right]\hat{p}_t$$

$$\hat{q}_t = \hat{z}_t + S_K\hat{k}_{t-1} + S_L\hat{l}_t + S_M\hat{\mathbb{M}}_t$$

$$\hat{p}_t = \beta\left(I-\delta_C\right)\hat{p}_{t+1} - \frac{1}{\sigma}\left(I - \beta\left(I-\delta_C\right)\right)\left[I + S_I^C\left(\sigma-1\right)\right]\hat{c}_t$$

Step 3: Use the law of motion for capital to re-write the resource constraint as $\hat{\mathbb{X}}_t = \frac{1}{\delta_K}\hat{k}_t - \frac{1-\delta_K}{\delta_K}\hat{k}_{t-1}$ and simplify:

$$\left(\frac{1}{\phi}S_L^L+I\right)\hat{l}_t = \hat{p}_t + \hat{q}_t$$

$$S_1^X\hat{p}_t = \left[S_1^X\beta(1-\delta_K) + \tilde{\beta}\right]\mathbb{E}_t[\hat{p}_{t+1}] + \tilde{\beta}\left[\mathbb{E}_t[\hat{q}_{t+1}] - \hat{k}_t\right]$$

$$\left[I - \tilde{S}_M^Q T_1\right]\hat{q}_t = (\delta_C)^{-1}\tilde{S}_C^Q\hat{c}_t - \left((\delta_C)^{-1} - I\right)\tilde{S}_C^Q\hat{c}_{t-1} + \frac{1}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_t -$$

$$\frac{1-\delta_K}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_{t-1}+\left[\tilde{S}_M^Q T_1-\tilde{S}_M^Q T_2-\tilde{S_X^Q}T_2+\tilde{S}_X^Q T_1 S_1^X\right]\hat{p}_t$$

$$\hat{q}_t=\hat{z}_t+S_K\hat{k}_{t-1}+S_L\hat{l}_t+S_M\hat{\mathbb{M}}_t$$

$$\hat{p}_t=\beta\left(I-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\sigma}\left(I-\beta\left(I-\delta_C\right)\right)\left[I+S_I^C\left(\sigma-1\right)\right]\hat{c}_t$$

Step 4: Re-write the production function using $\hat{\mathbb{M}}_t=\hat{q}_t+\left(I-\Gamma'_M\right)\hat{p}_t$

$$\left(\frac{1}{\phi}S_L^L+I\right)\hat{l}_t=\hat{p}_t+\hat{q}_t$$

$$S_1^X\hat{p}_t=\left[S_1^X\beta(1-\delta_K)+\tilde{\beta}\right]\mathbb{E}_t[\hat{p}_{t+1}]+\tilde{\beta}\left[\mathbb{E}_t[\hat{q}_{t+1}]-\hat{k}_t\right]$$

$$\left[I-\tilde{S}_M^Q T_1\right]\hat{q}_t=\left(\delta_C\right)^{-1}\tilde{S}_C^Q\hat{c}_t-\left(\left(\delta_C\right)^{-1}-I\right)\tilde{S}_C^Q\hat{c}_{t-1}+\frac{1}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_t-$$

$$\frac{1-\delta_K}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_{t-1}+\left[\tilde{S}_M^Q T_1-\tilde{S}_M^Q T_2-\tilde{S_X^Q}T_2+\tilde{S}_X^Q T_1 S_1^X\right]\hat{p}_t$$

$$\left[I-S_M\right]\hat{q}_t=\hat{z}_t+S_K\hat{k}_{t-1}+S_L\hat{l}_t+S_M\left(I-\Gamma'_M\right)\hat{p}_t$$

$$\hat{p}_t=\beta\left(I-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\sigma}\left(I-\beta\left(I-\delta_C\right)\right)\left[I+S_I^C\left(\sigma-1\right)\right]\hat{c}_t$$

Step 5: Re-write the production function and the investment Euler equation using that optimality implies:

$$\hat{l}_t=\left(\frac{1}{\phi}S_L^L+I\right)^{-1}\hat{p}_t+\left(\frac{1}{\phi}S_L^L+I\right)^{-1}\hat{q}_t$$

$$S_1^X\hat{p}_t=\left[S_1^X\beta(1-\delta_K)+\tilde{\beta}\right]\mathbb{E}_t[\hat{p}_{t+1}]+\tilde{\beta}\left[\mathbb{E}_t[\hat{q}_{t+1}]-\hat{k}_t\right]$$

$$\left[I-\tilde{S}_M^Q T_1\right]\hat{q}_t=\left(\delta_C\right)^{-1}\tilde{S}_C^Q\hat{c}_t-\left(\left(\delta_C\right)^{-1}-I\right)\tilde{S}_C^Q\hat{c}_{t-1}+\frac{1}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_t-$$

$$\frac{1-\delta_K}{\delta_K}\tilde{S_X^Q}T_1\hat{k}_{t-1}+\left[\tilde{S}_M^Q T_1-\tilde{S}_M^Q T_2-\tilde{S_X^Q}T_2+\tilde{S}_X^Q T_1 S_1^X\right]\hat{p}_t$$

$$\underbrace{\left[\left[I-S_M\right]-S_L\left(\frac{1}{\phi}S_L^L+I\right)^{-1}\right]}_{\Lambda^{-1}}\hat{q}_t=\hat{z}_t+S_K\hat{k}_{t-1}+\left[S_M\left(I-\Gamma'_M\right)+S_L\left(I+\frac{1}{\phi}S_L^L\right)^{-1}\right]\hat{p}_t$$

$$\hat{p}_t=\beta\left(I-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\sigma}\left(I-\beta\left(I-\delta_C\right)\right)\left[I+S_I^C\left(\sigma-1\right)\right]\hat{c}_t$$

Step 6: Use $\hat{q}_t=\Lambda\hat{z}_t+\Lambda S_K\hat{k}_{t-1}+\Lambda\left[S_M\left(I-\Gamma'_M\right)+S_L\left(\frac{1}{\phi}S_L^L+I\right)^{-1}\right]\hat{p}_t$

$$0 = -S_1^X \hat{p}_t + \left[S_1^X \beta(1-\delta_K) + \tilde{\beta} + \tilde{\beta}\Lambda \left[S_M \left(I - \Gamma'_M\right) + S_L \left(\frac{1}{\phi} S_L^L + I \right)^{-1} \right] \right] \mathbb{E}_t[\hat{p}_{t+1}]$$

$$+ \tilde{\beta}\Lambda \mathbb{E}_t \left[\hat{z}_{t+1}\right] + \tilde{\beta} \left[\Lambda S_K - I\right] \hat{k}_t$$

$$0 = \left(\delta_C\right)^{-1} \tilde{S}_C^Q \hat{c}_t - \left(\left(\delta_C\right)^{-1} - I\right) \tilde{S}_C^Q \hat{c}_{t-1} + \frac{1}{\delta_K} \tilde{S_X^Q} T_1 \hat{k}_t$$

$$- \left[\frac{1-\delta_K}{\delta_K} \tilde{S_X^Q} T_1 + \left[I - \tilde{S}_M^Q T_1 \right] \Lambda S_K \right] \hat{k}_{t-1} - \left[I - \tilde{S}_M^Q T_1 \right] \Lambda \hat{z}_t$$

$$+ \left[\tilde{S}_M^Q T_1 - \tilde{S}_M^Q T_2 - \tilde{S_X^Q} T_2 - \left[I - \tilde{S}_M^Q T_1 \right] \Lambda \left[S_M \left(I - \Gamma'_M\right) + S_L \left(\frac{1}{\phi} S_L^L + I \right)^{-1} \right] + \tilde{S}_X^Q T_1 S_1^X \right] \hat{p}_t$$

$$0 = -\hat{p}_t + \beta \left(I - \delta_C\right) \hat{p}_{t+1} - \frac{1}{\sigma} \left(I - \beta\left(I - \delta_C\right)\right) \left[I + S_I^C \left(\sigma - 1\right) \right] \hat{c}_t$$

Step 7 Stack Equations:
Call $\tilde{\Lambda} \equiv \Lambda \left[S_M \left(I - \Gamma'_M\right) + S_L \left(\frac{1}{\phi} S_L^L + I \right)^{-1} \right]$

$$0 = \begin{bmatrix} S_1^X \beta(1-\delta_K) + \tilde{\beta} + \tilde{\beta}\tilde{\Lambda} & \tilde{\beta}\left[\Lambda S_K - I\right] & 0 \\ 0 & \frac{1}{\delta_K} \tilde{S_X^Q} T_1 & \left(\delta_C\right)^{-1} \tilde{S}_C^Q \\ \beta\left(I - \delta_C\right) & 0 & -\frac{1}{\sigma}\left(I - \beta\left(I - \delta_C\right)\right)\left[I + S_I^C\left(\sigma - 1\right)\right] \end{bmatrix} \begin{bmatrix} \mathbb{E}_t[\hat{p}_{t+1}] \\ \hat{k}_t \\ \hat{c}_t \end{bmatrix}$$

$$+ \begin{bmatrix} -S_1^X & 0 & 0 \\ \tilde{S}_M^Q T_1 - \tilde{S}_M^Q T_2 - \tilde{S_X^Q} T_2 - \left[I - \tilde{S}_M^Q T_1 \right] \tilde{\Lambda} + \tilde{S}_X^Q T_1 S_1^X & -\left[\frac{1-\delta_K}{\delta_K} \tilde{S_X^Q} T_1 + \left[I - \tilde{S}_M^Q T_1 \right] \Lambda S_K \right] & -\left(\left(\delta_C\right)^{-1} - I\right) \\ -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{p}_t \\ \hat{k}_{t-1} \\ \hat{c}_{t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{\beta}\Lambda \\ -\left[I - \tilde{S}_M^Q T_1 \right] \Lambda \\ 0 \end{bmatrix} \hat{z}_t$$

$$\frac{1}{\delta_K} \tilde{S_X^Q} T_1 \hat{k}_t + \left(\delta_C\right)^{-1} \tilde{S}_C^Q \hat{c}_t +$$

$$\hat{P}_t = -\frac{1}{\sigma} \left[I + \sum_{I=1}^{N} \left(\sigma - 1\right)\left(1 - b\right) S_I^C \right] \hat{c}_t$$

With no consumption good durability:

$$0 = \begin{bmatrix} S_1^X \beta(1-\delta_K) + \tilde{\beta} + \tilde{\beta}\tilde{\Lambda} & \tilde{\beta}\left[\Lambda S_K - I\right] \\ 0 & \frac{1}{\delta_K} \tilde{S_X^Q} T_1 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t[\hat{p}_{t+1}] \\ \hat{k}_t \end{bmatrix}$$

$$+ \begin{bmatrix} -S_1^X & 0 \\ \tilde{S}_M^Q T_1 - \tilde{S}_M^Q T_2 - \tilde{S_X^Q} T_2 - \left[I - \tilde{S}_M^Q T_1 \right] \tilde{\Lambda} + \tilde{S}_X^Q T_1 S_1^X - \frac{1}{\sigma}\left[I + S_I^C\left(\sigma - 1\right)\right] \tilde{S}_C^Q & -\left[\frac{1-\delta_K}{\delta_K} \tilde{S_X^Q} T_1 + \left[I - \tilde{S}_M^Q T_1 \right] \Lambda S_K \right] \end{bmatrix}$$

$$\times \begin{bmatrix} \hat{p}_t \\ \hat{k}_{t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\beta}\Lambda \\ -\left[I - \tilde{S}_M^Q T_1\right]\Lambda \end{bmatrix} \hat{z}_t$$

A.5 Blanchard-Kahn

We have expressed the reduced system as

$$\begin{bmatrix} \mathbb{E}_t[\hat{p}_{t+1}] \\ \hat{k}_t \\ E_\delta^C \hat{c}_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} \hat{p}_t \\ \hat{k}_{t-1} \\ E_\delta^C \hat{c}_{t-1} \end{bmatrix} + \mathbf{B}\hat{z}_t,$$

where $\mathbb{A}$ has $N + N_\delta^C$ stable eigenvalues and N unstable eigenvalues.
Using a Jordan decomposition write $A = VDV^{-1}$ where D is diagonal and is ordered such that the N explosive eigenvalues are ordered first and the $N + N_\delta^C$ stable eigenvalues are ordered last. Re-write:

$$\mathbb{E}_t[Y_{t+1}] \equiv V^{-1} \begin{bmatrix} \mathbb{E}_t[\hat{p}_{t+1}] \\ \hat{k}_t \\ E_\delta^C \hat{c}_t \end{bmatrix} = DV^{-1} \begin{bmatrix} \hat{p}_t \\ \hat{k}_{t-1} \\ E_\delta^C \hat{c}_{t-1} \end{bmatrix} + V^{-1}\mathbf{B}\hat{z}_t, \equiv DY_t + \tilde{\mathbf{B}}\hat{z}_t,$$

Partition Y_t into the first $N \times 1$ block $Y_{1,t}$ and the lower $(N + N_\delta^C) \times 1$ block $Y_{2,t}$ and similarly for D and $\tilde{\mathbf{B}}$. Re-write:

$$Y_{1,t} = D_1^{-1}\mathbb{E}_t[Y_{1,t+1}] - D_1^{-1}\tilde{\mathbf{B}}_1 z_t$$

Substitute recursively

$$Y_{1,t} = -D_1^{-1} \sum_{s=0}^{T-1} D_1^{-s}\tilde{\mathbf{B}}_1 \mathbb{E}_t[z_{t+s}] + D_1^{-T}\mathbb{E}_t[Y_{1,t+1}] \to -D_1^{-1}(I - D_1^{-1})^{-1}\tilde{\mathbf{B}}_1 z_t$$

using the random walk assumption.
For $Y_{2,t}$ simply:

$$Y_{2,t} = D_2 Y_{2,t-1} + \tilde{\mathbf{B}}_2 z_t.$$

Note that

$$\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} = V^{-1} \begin{bmatrix} \hat{p}_t \\ \hat{k}_{t-1} \\ E_\delta^C \hat{c}_{t-1} \end{bmatrix}$$

and therefore, from the first set of N equations:

$$\begin{aligned}\hat{p}_t &= -(V_{11}^{-1})^{-1}V_{12}^{-1}\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} + (V_{11}^{-1})^{-1}Y_{1,t}\\ &= -(V_{11}^{-1})^{-1}V_{12}^{-1}\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} - (V_{11}^{-1})^{-1}D_1^{-1}(I-D_1^{-1})^{-1}\tilde{\mathbf{B}}_1 z_t\end{aligned}$$

The endogenous state evolves as follows:

$$\begin{aligned}\begin{bmatrix}\hat{k}_t\\ E_\delta^C\hat{c}_t\end{bmatrix} &= \mathbf{A}_{22}\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} + \mathbf{A}_{21}\hat{p}_t + \mathbf{B}_2 z_t\\ &= (\mathbf{A}_{22} - \mathbf{A}_{21}(V_{11}^{-1})^{-1}V_{12}^{-1})\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} - \mathbf{A}_{21}(V_{11}^{-1})^{-1}D_1^{-1}(I-D_1^{-1})^{-1}\tilde{\mathbf{B}}_1 z_t + \mathbf{B}_2 z_t\end{aligned} \tag{32}$$

For future reference, in the case without durable consumption:

$$\hat{p}_t = \mathbf{A}_{21}^{-1}\hat{k}_t - \mathbf{A}_{21}^{-1}\mathbf{A}_{22}\hat{k}_{t-1} - \mathbf{A}_{21}^{-1}\mathbf{B}_2 z_t$$

In the general case with durable consumption, pre-multiply (32) with the transpose of the $(n_k + n_c) \times n_k$ matrix $\mathbf{A}_{21}$:

$$\mathbf{A}_{21}'\begin{bmatrix}\hat{k}_t\\ E_\delta^C\hat{c}_t\end{bmatrix} = \mathbf{A}_{21}'\mathbf{A}_{22}\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} + \mathbf{A}_{21}'\mathbf{A}_{21}\hat{p}_t + \mathbf{A}_{21}'\mathbf{B}_2 z_t$$

Note that if $\mathbf{A}_{21}$ has full column rank, then $\mathbf{A}_{21}'\mathbf{A}_{21}$ is a square, non-singular matrix. Hence:

$$\hat{p}_t = (\mathbf{A}_{21}'\mathbf{A}_{21})^{-1}\mathbf{A}_{21}'\begin{bmatrix}\hat{k}_t\\ E_\delta^C\hat{c}_t\end{bmatrix} - (\mathbf{A}_{21}'\mathbf{A}_{21})^{-1}\mathbf{A}_{21}'\mathbf{A}_{22}\begin{bmatrix}\hat{k}_{t-1}\\ E_\delta^C\hat{c}_{t-1}\end{bmatrix} - (\mathbf{A}_{21}'\mathbf{A}_{21})^{-1}\mathbf{A}_{21}'\mathbf{B}_2 z_t \tag{33}$$

A.6 Filtering

Define

$$\tilde{V} \equiv (V_{11}^{-1})^{-1}D_1^{-1}(I-D_1^{-1})^{-1}\tilde{\mathbf{B}}_1$$

$$\tilde{S} \equiv \left(\frac{1}{\phi} S_L^L + I\right)^{-1}$$

In the case with no consumption good durability:

$$\begin{aligned}
\hat{l}_t &= \tilde{S}\hat{p}_t + \tilde{S}\hat{q}_t \\
&= \tilde{S}\hat{p}_t + \tilde{S}\left[\Lambda\hat{z}_t + \Lambda S_K \hat{k}_{t-1} + \Lambda\left[S_M\left(I - \Gamma_M'\right) + S_L\tilde{S}\right]\hat{p}_t\right] \\
&= \underbrace{\tilde{S}\left[I + \Lambda\left[S_M\left(I - \Gamma_M'\right) + S_L\tilde{S}\right]\right]}_{\Pi_{PL}}\hat{p}_t + \tilde{S}\Lambda\hat{z}_t + \tilde{S}\Lambda S_K\hat{k}_{t-1} \\
&= \left[-\Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{A}_{22} + \tilde{S}\Lambda S_K\right]\hat{k}_{t-1} + \Pi_{PL}\mathbf{A}_{21}^{-1}\hat{k}_t + \left[-\Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{B}_2 + \tilde{S}\Lambda\right]\hat{z}_t \\
&= \left[-\Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{A}_{22} + \tilde{S}\Lambda S_K\right]\hat{k}_{t-1} + \left[-\Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{B}_2 + \tilde{S}\Lambda\right]\hat{z}_t \\
&+ \left[\Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{A}_{22} - \Pi_{PL}(V_{11}^{-1})^{-1}V_{12}^{-1}\right]\hat{k}_{t-1} + \left[-\Pi_{PL}\tilde{V} + \Pi_{PL}\mathbf{A}_{21}^{-1}\mathbf{B}_2\right]z_t \\
&= \underbrace{\left[\tilde{S}\Lambda S_K - \Pi_{PL}(V_{11}^{-1})^{-1}V_{12}^{-1})\right]}_{\Pi_k}\hat{k}_{t-1} + \underbrace{\left[\tilde{S}\Lambda - \Pi_{PL}\tilde{V}\right]}_{\Pi_z}\hat{z}_t
\end{aligned}$$

$$\begin{aligned}
k_{t-1} &= \Pi_k^{-1}\hat{l}_t - \Pi_k^{-1}\Pi_z\hat{z}_t \\
\hat{l}_{t+1} &= \Pi_k\hat{k}_t + \Pi_z\hat{z}_{t+1} \\
&= \Pi_k\left[\mathbf{A}_{22} - \mathbf{A}_{21}(V_{11}^{-1})^{-1}V_{12}^{-1}\right]\hat{k}_{t-1} + \Pi_z\hat{z}_{t+1} + \Pi_k\left[\mathbf{B}_2 - \mathbf{A}_{21}\tilde{V}\right]z_t \\
&= \underbrace{\Pi_k\left[(\mathbf{A}_{22} - \mathbf{A}_{21}(V_{11}^{-1})^{-1}V_{12}^{-1})\right]\Pi_k^{-1}}_{\equiv\varrho}\hat{l}_t + \Pi_z\hat{z}_{t+1} \\
&+ \underbrace{\Pi_k\left[\mathbf{B}_2 - \mathbf{A}_{21}\tilde{V} + \left[\mathbf{A}_{22} - \mathbf{A}_{21}(V_{11}^{-1})^{-1}V_{12}^{-1}\right]\Pi_k^{-1}\Pi_z\right]}_{\equiv\Xi}z_t
\end{aligned}$$

B Data with and without detrending

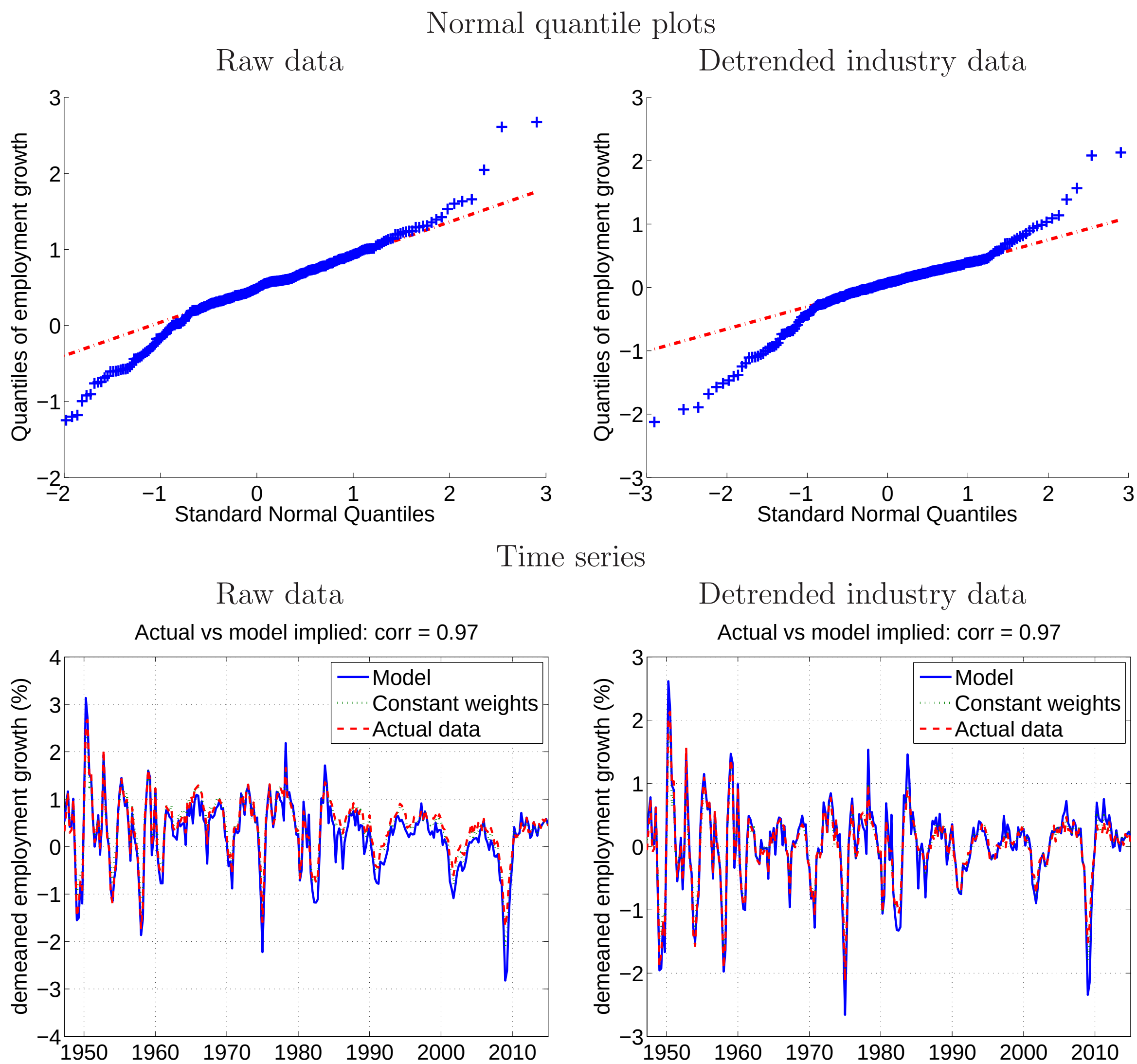

Figure 3: Comparing aggregate employment growth with and without detrending at the industry level

C Quality of the approximation

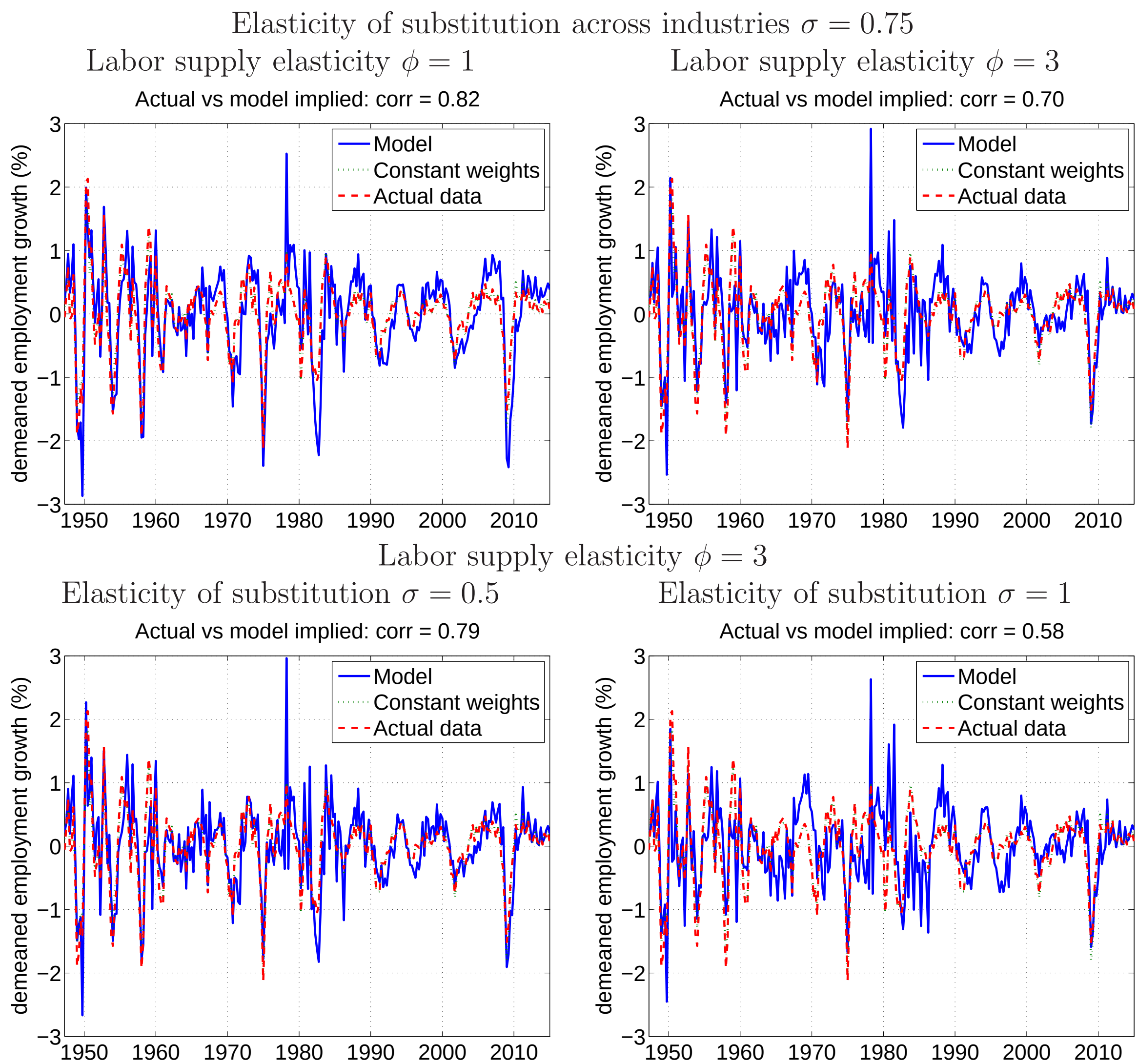

Figure 4: Model-implied employment based on cross-industry average of filtered productivty shocksversus actual employment for different preference parameters.

D Implied productivity shocks

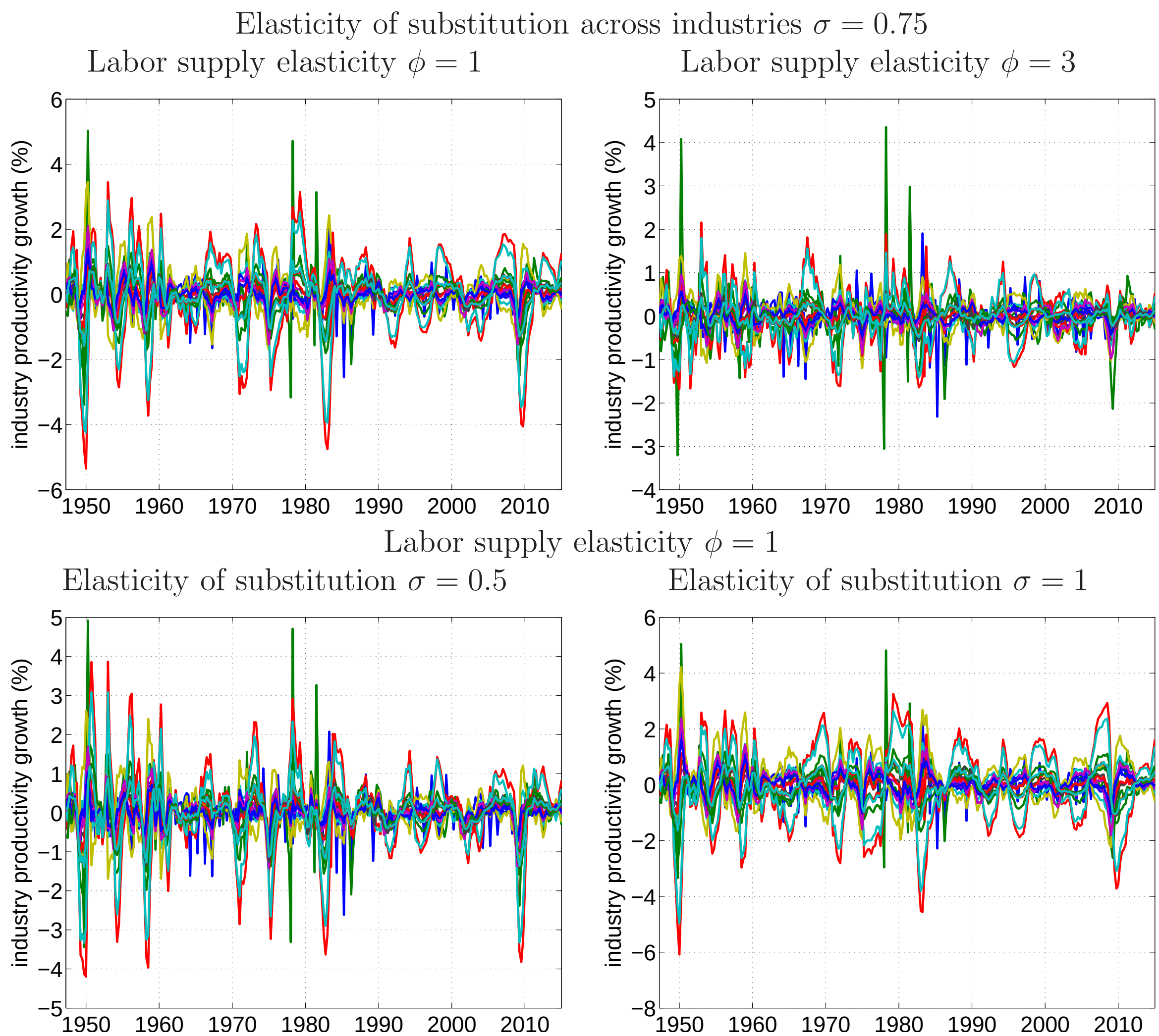

Figure 5: Model-implied productivity shocks for all industries for different preference parameters

E Historical decomposition

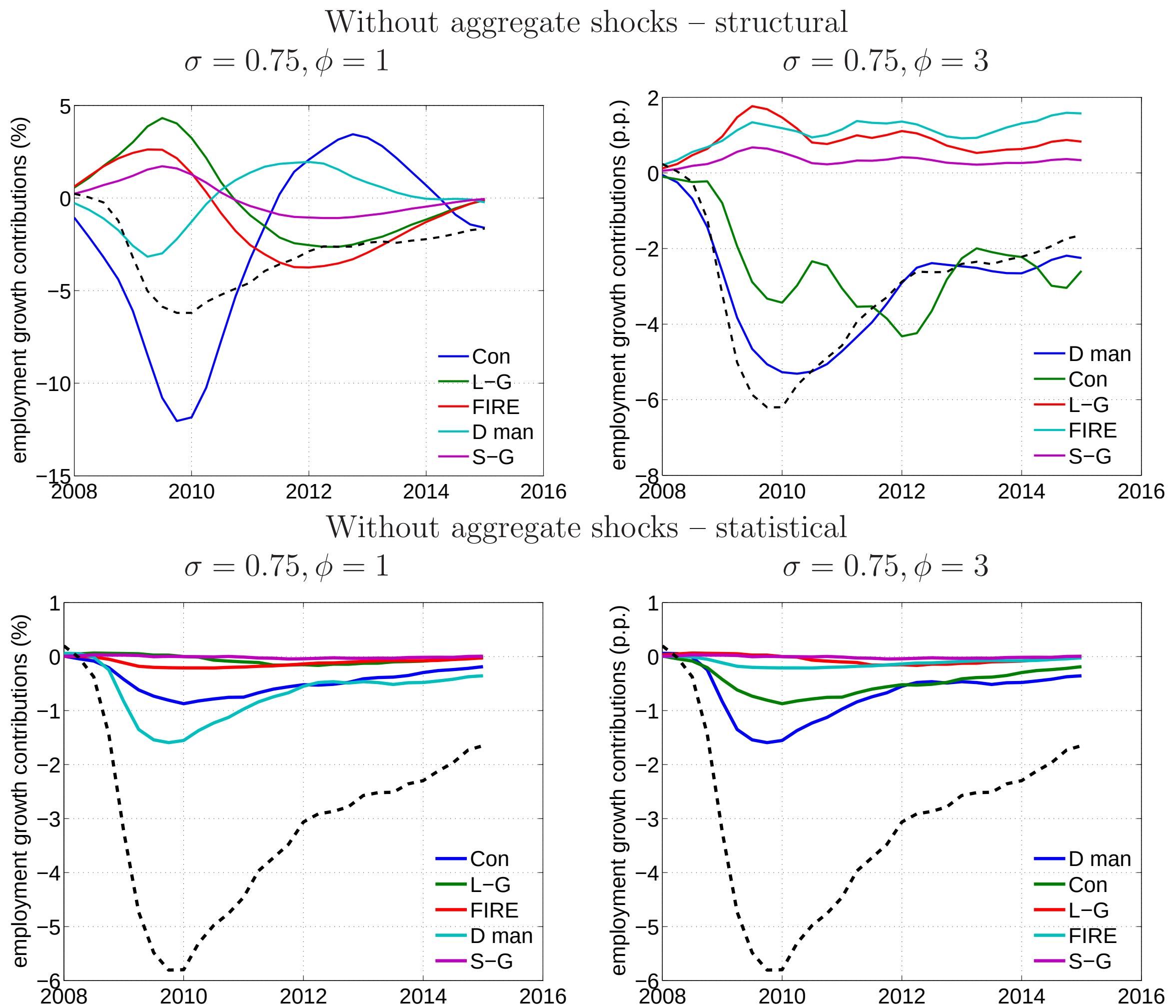

Figure 6: Cumulative historical employment in Great Recession without aggregate component – top 5 industries with structural contribution in 2010

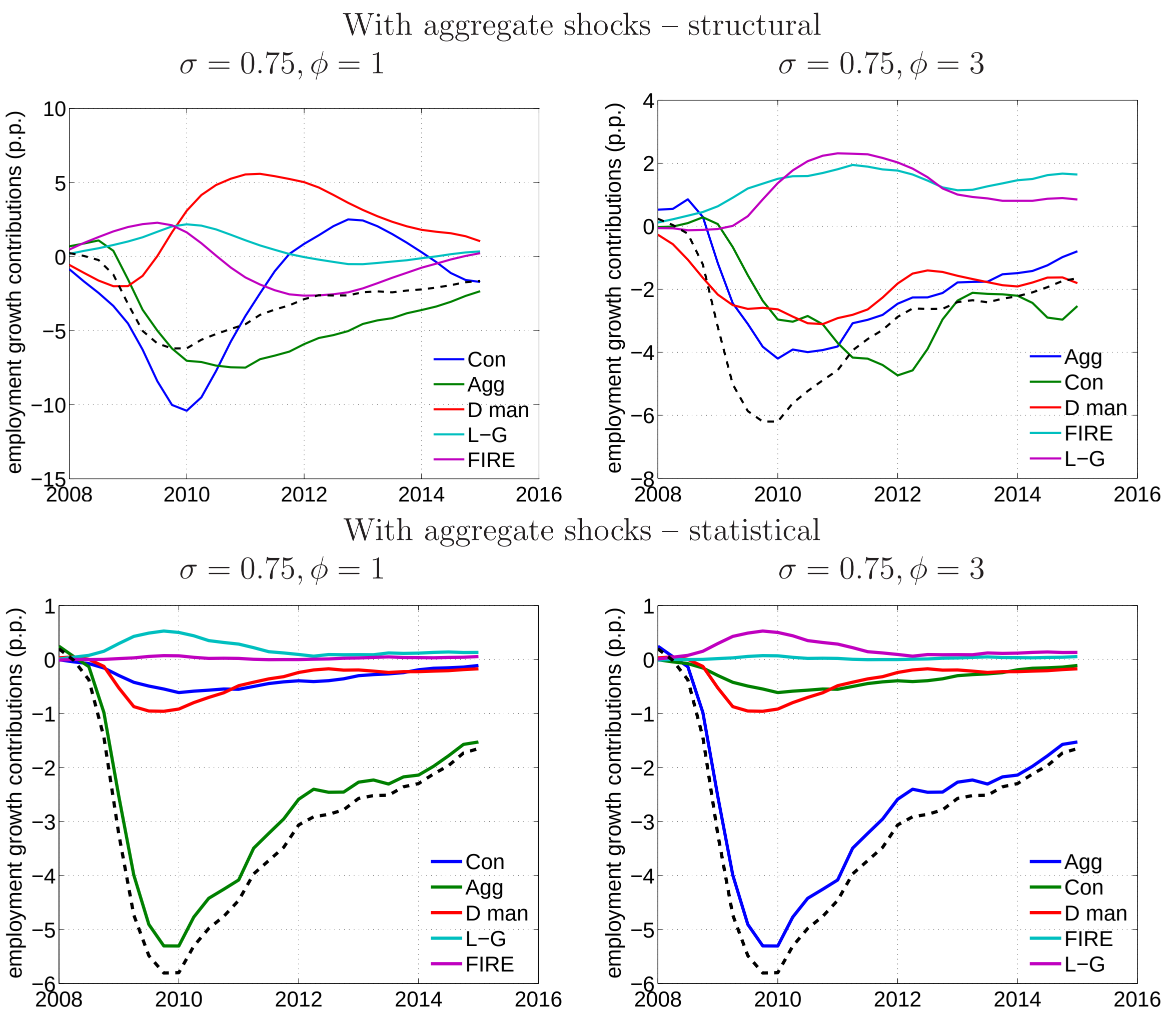

Figure 7: Cumulative historical employment in Great Recession with aggregate component – top 5 industries with structural contribution in 2010